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THE ENUMERATIVE GEOMETRY OF PLANE CUBICS. I: SMOOTH CUBICS

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ABSTRACT. We construct a variety of complete plane cubics by a sequence of five blow-ups over $\mathbf{P}^9$. This enables us to translate the problem of computing characteristic numbers for a family of plane cubics into one of computing five Segre classes, and to recover classic enumerative results of Zeuthen and Maillard.

0 INTRODUCTION

This paper is devoted to the computation of the *characteristic numbers* for the 9-dimensional family of smooth plane cubics, i.e. the number of nonsingular plane cubics which are tangent to n_l lines and contain $9 - n_l$ points in general position in the plane. We plan to complement this result with the computation of the characteristic numbers for nodal and cuspidal cubics, in a forthcoming paper.

Classically, the enumerative geometry of plane cubics was studied independently by S. Maillard and H. S. Zeuthen around 1870 [M, Z]; their results also appear in [Sc, Chapter 4, §24]. However, as with many other accomplishments of the great enumerative geometers of the nineteenth century, the rigor of the methods used in [M, Z] was soon questioned. In the past few years interest in enumerative geometry has revived, partially as a consequence of a new and deeper understanding of intersection theory, and in many cases the old results and methods have been verified, improved, or corrected. In the case of plane cubics, there are already several modern approaches partially verifying Maillard and Zeuthen's results (see [Sa, KS, XM]). In general, these approaches work in the vein of the classic "degeneration method": by specializing the families to more degenerate ones, and using previously obtained results. Kleiman and Speiser, in particular, have developed an efficient procedure of "partially" compactifying the family under examination. They normalize *part* of the graph of the dual map, in such a way that the elementary systems (the basic tool for relating the characteristic numbers of different families) are incorporated as complete subschemes.

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We present here an approach with a different flavor. We dominate the graph of the dual map with a nonsingular variety $\widetilde{V}$, which is obtained from the $\mathbb{P}^9$ parametrizing plane cubics by a sequence of blow-ups, in the spirit of (for example) Veinsencher's spaces of 'complete quadrics' [V]. Also, rather than employing the method of degeneration, we reduce the computation of the numbers for a family of reduced cubics to the computation of certain Segre classes related to the behavior of the family in the blow-up process. This choice, which has its roots in Fulton-Mac Pherson's 'static' intersection theory, forces us to an extensive analysis of the blow-ups —indeed, this choice forces us to desingularize the *whole* graph; its net advantage is that a specific family can be studied without dependence on other results. The characteristic numbers for smooth cubics, for example, are an immediate by-product of our construction of $\widetilde{V}$, while their computation via the degeneration method relies on sophisticated information about families of *singular* cubics.

The compactification we construct here can be used to compute characteristic numbers for families of singular cubics as well; generally speaking, the difficulty of the task increases with the *co*dimension of the family, in contrast with the degeneration method.

We give a sequence of 5 blow-ups $V_1, \dots, V_5$ over $\mathbb{P}^9 = V_0$, with nonsingular centers $B_i \hookrightarrow V_i$; if $F \subset \mathbb{P}^9$ is a family of reduced cubics, and $F_1, \dots F_5$ are the proper transforms in the blow-ups of its closure F_0 in $\mathbb{P}^9$, we basically translate the problem of computing the characteristic numbers for F into one of computing the five Segre classes $s(B_i \cap F_i, F_i)$, $i = 0, \dots, 4$. Now, in general these are easier to compute when the codimension of F is low; for F the family of smooth cubics, $F_i = V_i$ and $s(B_i, V_i)$ are the inverse Chern classes of the normal bundles to B_i in V_i, which are obtained in the blow-up construction. The classes needed when F parametrizes other families (e.g. nodal cubics, or cuspidal cubics, or cubics tangent to a line at a given point) will require some more work.

In the $\mathbb{P}^9$ parametrizing plane cubic curves, call 'point-conditions' and 'line-conditions' respectively the hypersurfaces consisting of the cubics respectively containing a given point and tangent to a given line. The intersection of all line-conditions in $\mathbb{P}^9$ is supported on a four-dimensional irreducible variety S parametrizing all nonreduced cubics—i.e., cubics decomposing into a line and a 'double line'. For any family of *reduced* cubics $F \subset \mathbb{P}^9 - S$, consider the number N of elements (counted with multiplicity) in the intersection of F with given general point- and line-conditions. For example, if F is the set of all *smooth* cubics, N is a characteristic number for smooth cubics.

Now, for any variety mapping to $\mathbb{P}^9$, isomorphically over $\mathbb{P}^9 - S$, call 'point-' and 'line-conditions' the proper transforms of the conditions in $\mathbb{P}^9$; we say that such a variety $\widetilde{V}$ is a 'variety of complete plane cubics' if the intersection of its line-conditions is *empty*. In §1 we prove (Theorem I) that the number N is precisely the degree of the intersection of the point- and line-conditions in such a $\widetilde{V}$ with the proper transform $\widetilde{F} \subset \widetilde{V}$ of the *closure* of F.

In §3 we construct a smooth variety $\widetilde{V}$ of complete cubics. This is obtained by a sequence of five blow-ups along nonsingular centers, starting with the blow-up of the $\mathbb{P}^9$ of cubics along the Veronese of 'triple lines'. The same sequence was considered by U. Sterz, who also obtains some enumerative results (see in particular [St, IV]), and to which we address the reader for a different point of view. The general aim is to separate the proper transforms of the line-conditions above S; we accomplish this by systematically blowing up the largest component of their intersection. In doing so, we also collect (Theorem III) the information required to compute in $\widetilde{V}$ the intersection degrees we need: i.e., a description of the intersection rings of the centers of the blow-ups, the total Chern classes of their normal bundles, and information consisting essentially of the multiplicities of the conditions along the centers.

The computation of the intersection degrees is performed by using a formula (Theorem II in §2) which relates intersections under blow-ups. For X_ν subschemes of a scheme V, and $\widetilde{V}$ the blow-up of V along a regularly imbedded subscheme B, the formula gives the difference between the intersection number of the X_ν in V and the intersection number of their proper transforms in $\widetilde{V}$ explicitly, in terms of information essentially equivalent to the Segre classes $s(B \cap X_\nu, X_\nu)$. We can apply this formula to climb the sequence of blow-ups defining our variety of complete cubics.

In view of Theorems I, II, and III, the key information for computing the characteristic numbers for any family F of cubics amounts to five Segre classes $s(B_i \cap F_i, F_i)$, where F_i are the proper transforms of the closure of F in $\mathbb{P}^9$. In fact, this result is best expressed in terms of equivalent data, i.e. the 'full intersection classes'

$$B_i \circ F_i = c(N_{B_i} V_i) \cap s(B_i \cap F_i, F_i).$$

Theorem IV gives the numbers for a family F of reduced cubics explicitly in terms of the classes $B_i \circ F_i$. For the family F of all smooth cubics we have $F_i = V_i$, thus $B_i \circ F_i = B_i$ (since, for B, V smooth, the Segre class $s(B, V)$ equals the inverse total Chern class $c(N_B V)^{-1}$). This allows us to get the characteristic numbers for smooth cubics by simply evaluating coefficients of certain power series (Corollary IV).

For F the family of nodal cubics, or of cuspidal cubics, etc., the computation of the classes $B_i \circ F_i$ is a more challenging task. We will devote to it a second note.

A good example of a less trivial application of Theorem IV to smooth cubics is the computation of the characteristic numbers obtained by considering also the codimension-2 condition expressing the tangency to a line *at a given point.* To apply Theorem IV to this question, we have to compute the five classes for the family of cubics satisfying one of these conditions. This computation is sketched in §5; the characteristic numbers (agreeing with Maillard and Zeuthen's results) are listed in Corollary IV$'$. In fact, we show that the information we need to compute the numbers with respect to codimension-1 conditions for *any* family of cubics (i.e. the five classes) is enough to obtain the results involving these codimension-2 conditions as well (Theorem IV$'$ in §5). This result will also be applied to families of singular cubics in the future note.

In this paper we work over an algebraically closed field of characteristic $\neq 2, 3$. The blow-up formula in §2 is characteristic-free, and the preliminary results (in particular Corollary I) hold in characteristic $\neq 2$; however, the blow-up construction for the space of cubics needs characteristic $\neq 2, 3$.

Some of the material in this paper appears in the author's doctoral thesis written under the guidance of W. Fulton at Brown (May 1987), and (in a sketchier version) in [A].

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1 Preliminaries: varieties of complete plane curves

We will discuss here some facts and notations we will use in the rest of the note. The facts hold for any degree and any family of reduced curves, so we will not restrict ourselves to smooth cubics.

In the $\mathbb{P}^N$ parametrizing plane curves of degree d, call *point-conditions* and *line-conditions* respectively the hypersurfaces consisting of the plane curves respectively containing a given point and tangent to a given line. By 'tangent to a line' we will always mean 'intersecting a line with multiplicity at least 2 at a point'. We will say that a curve c and a line l are '*properly* tangent' if l is simply tangent to c at a single nonsingular point—i.e., if the tangency point is smooth on c and c is a smooth point of the line-condition corresponding to l.

For any variety $\widetilde{V}$ mapping birationally to $\mathbb{P}^N$, biregularly over the set $\mathbb{P}^N - S$ consisting of reduced curves, call the proper transforms of the point- and the line-conditions of $\mathbb{P}^N$ *point-* and *line-conditions of* $\widetilde{V}$.

Definition. We shall say that $\widetilde{V}$ is a *variety of complete plane curves of degree* d if, moreover, the intersection of all its line-conditions is empty.

The general point- and line-conditions of $\widetilde{V}$ define divisors $\widetilde{P}, \widetilde{L}$ in $\widetilde{V}$. Although in general these need not be Cartier divisors on $\widetilde{V}$, notice that they restrict to Cartier divisors on the inverse image of $\mathbb{P}^N - S$: thus if their intersection with a subvariety $\widetilde{F}$ of $\widetilde{V}$ is proper and does not have components lying over S, then intersection products $\widetilde{P} \cdot \widetilde{F}$ and $\widetilde{L} \cdot \widetilde{F}$ are defined. When writing such products, we will imply that this is the case.

Our aim in this section is to show

Theorem I. *Let $\widetilde{V}$ be a variety of complete plane curves of degree d, F an r-dimensional (maybe noncomplete) subvariety in $\mathbb{P}^N$ parametrizing a family of reduced curves, and let $\widetilde{F}$ be the proper transform in $\widetilde{V}$ of the closure of F. Then the number of elements (counted with multiplicities) of F containing n_p given points and tangent to n_l given lines in general position, with $n_p + n_l = r$, is $\widetilde{P}^{n_p} \cdot \widetilde{L}^{n_l} \cdot \widetilde{F}$. Furthermore, the elements containing the given points and properly tangent to the given lines are counted with multiplicity* 1.

Note that the statement implies that this number does not change when F is replaced with any dense open subset of F. I.e., 'special' curves in the family can be discarded.

In this note, our main application of this result is to the computation of the characteristic numbers for the family of smooth plane cubics. Since in characteristic $\neq 2$ the general smooth curve is reflexive (so that for general lines all tangencies will be proper), Theorem I gives

Corollary I. *The characteristic numbers for the family of smooth plane curves of degree d are given by $\widetilde{P}^{n_p} \cdot \widetilde{L}^{n_l}$, for all n_p, n_l with $n_p + n_l = \frac{d(d+3)}{2}$.*

In §3 we will construct a 'variety of complete plane cubics'; Corollary I will then allow us to explicitly perform the computation for smooth plane cubics. More generally, Theorem I and the construction in §3 will give a tool (Theorem IV in §4) to compute the numbers for any family of reduced cubics, on the basis of geometric information.

Let Q be a 3-dimensional vector space over an algebraically closed field of characteristic $\neq 2$. The curves of degree d in the projective plane $\mathbb{P}^2 = \mathbb{P}(Q)$ form a projective space $\mathbb{P}^N = \mathbb{P}(\mathrm{Sym}^d \check{Q})$, of dimension $N = \frac{d(d+3)}{2}$. In this projective space, the curves that contain a given point form a hyperplane; while those that are *tangent* to a given line form a hypersurface of degree $2d - 2$. We will call these divisors *point-conditions* and *line-conditions* respectively. As a point varies in $\mathbb{P}^2$, the corresponding point-condition traces a subset of $\check{\mathbb{P}}^N$. In fact, if $(x_0 : x_1 : x_2)$ are coordinates in $\mathbb{P}^2$, the point-condition corresponding to the point $(\bar{x}_0 : \bar{x}_1 : \bar{x}_2)$ is the hyperplane in $\mathbb{P}^N$ whose equation has the monomials of degree d in $\bar{x}_0, \bar{x}_1, \bar{x}_2$ for coefficients. In other words:

Remark 1. The set of point-conditions of $\mathbb{P}^N$ is the d-Veronese imbedding of $\check{\mathbb{P}}^2$ in $\check{\mathbb{P}}^N$. In particular, the set of point-conditions is nondegenerate and irreducible, and in particular it is not contained in any finite union of hyperplanes: for example, it follows that a point-condition can always be chosen to cut properly finitely many arbitrary subvarieties of $\mathbb{P}^N$.

On the open subset of $\mathbb{P}^N$ formed by the *smooth* curves, an injective morphism is defined to the space $\mathbb{P}^M = \mathbb{P}(\mathrm{Sym}^{d(d-1)} \check{Q})$ parametrizing degree-$d(d-1)$ curves, by associating to each curve its *dual.* Note that a curve is tangent to a line $l \subset \mathbb{P}^2$ if and only if its dual contains $l \in \check{\mathbb{P}}^2$: thus the line-conditions in $\mathbb{P}^N$ are the pull-backs of the point-conditions in the $\mathbb{P}^M$ parametrizing all degree-$d(d-1)$ plane curves, and it follows (by Remark 1)

Remark 2. The set of line-conditions of $\mathbb{P}^N$ is the $d(d-1)$th Veronese imbedding of $\check{\mathbb{P}}^2$ into $\check{\mathbb{P}}^M$.

This also makes it clear that the rational map $\psi: \mathbb{P}^N \dashrightarrow \mathbb{P}^M$ determined by the morphism above is defined by the linear system generated by the line-conditions in $\mathbb{P}^N$.

We want to resolve the indeterminacies of ψ. These occur on the intersection of all line-conditions, supported on the variety $S \subset \mathbb{P}^N$ parametrizing *non-reduced* curves; ψ is an injective morphism on $\mathbb{P}^N - S$. We will call any variety $\widetilde{V}$ filling the commutative diagram

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\ \tilde{\psi}\ } & \mathbb{P}^M \\ {\scriptstyle\pi}\big\downarrow & & \big\| \\ \mathbb{P}^N & \overset{\psi}{\dashrightarrow} & \mathbb{P}^M \end{array}$$

with $\tilde{\psi}$ a morphism, and isomorphic to $\mathbb{P}^N$ outside $\pi^{-1}(S)$, a '*variety of complete plane curves of degree* d'. For example, the blow-up of $\mathbb{P}^N$ along the scheme-intersection of all its line-conditions is a variety of complete curves of degree d. An instance is the classical 'variety of complete conics' (cf. [CX, §2]). In a different contest, Vainsencher's varieties of complete quadrics (inspired by Schubert's work) give another example of a similar situation.

Note that the first condition (ψ lifting to a morphism $\widetilde{V} \to \mathbb{P}^M$) amounts to just requiring that the intersection in $\widetilde{V}$ of the proper transforms of all line-conditions be empty. We will construct a smooth variety of complete cubics by blowing-up $\mathbb{P}^9$ five times along suitable centers, and use this variety to compute the characteristic numbers of certain families of plane cubics.

The proper transforms of general point-conditions and line-conditions determine classes $\widetilde{P}, \widetilde{L}$ of divisors on $\widetilde{V}$. If $\widetilde{V}$ is smooth, then for any $\widetilde{F} \subset \widetilde{V}$ we have intersection products $\widetilde{P} \cdot \widetilde{F}$ and $\widetilde{L} \cdot \widetilde{F}$. Even if $\widetilde{V}$ is not smooth, however, $\widetilde{P}$ and $\widetilde{L}$ restrict to Cartier divisors on $\pi^{-1}(\mathbb{P}^N - S)$ (since this is smooth): thus $\widetilde{P} \cdot \widetilde{F}, \widetilde{L} \cdot \widetilde{F}$ are defined as long as $\pi(F) \not\subseteq S$ and the proper transforms of general choices of conditions cut $\widetilde{F} \cap \pi^{-1}S$ properly. This will always be the case for F as below (see Proposition 1 (1) and Lemma 1).

Computing the characteristic numbers for families of reduced plane curves amounts to computing the number of intersections of certain subsets of $\mathbb{P}^N$ with assortments of point- and line-conditions in general position.

Let us first consider line-conditions. Let F be a pure r-dimensional locally closed (maybe noncompact) subset of $\mathbb{P}^N$, parametrizing a family of reduced curves: i.e., we assume $F \cap S = \varnothing$. For example, F could be the set of all smooth curves, or the set of all *nodal* curves, or the set of all nodal curves containing a given point, and so on. The number of elements of F tangent to r lines in general position in the plane is the number of points of intersection of F with r general line-conditions of $\mathbb{P}^M$; but since all line-conditions contain the set of nonreduced curves, often nonreduced curves will appear in the intersection of r (general) line-conditions with the *closure* $\overline{F}$ of F. For example, if F is the set of nonsingular conics, the intersection of 5 general line-conditions with $\overline{F}$ (= the whole of $\mathbb{P}^5$) consists of one isolated point and of the 2-dimensional set of 'double lines'.

Let then $\widetilde{V}$ be a variety of complete plane curves of degree d, $\widetilde{F}$ the proper transform of $\overline{F}$ in $\widetilde{V}$, and call 'line-conditions in $\widetilde{V}$' the proper transforms in $\widetilde{V}$ of the line-conditions of $\mathbb{P}^N$; call $\widetilde{L}$ the class of the general line-condition in $\widetilde{V}$.

Proposition 1. *A line-condition in $\widetilde{V}$ can always be chosen to cut properly any finite collection of subvarieties of $\widetilde{V}$;*

With F as above, r line-conditions in $\widetilde{V}$ can be chosen to cut $\widetilde{F}$ in finitely many points, mapping to points of F; the number of elements of F that are tangent to r lines in general position is the number of intersections of $\widetilde{F}$ with r general line-conditions in $\widetilde{V}$.

Proof. (1) follows from Remark 2: the set of line-conditions is not contained in any finite union of hyperplanes of $\check{\mathbb{P}}^M$.

For (2), let π_F be the restriction of π to $\widetilde{F}$, and set $E = \overline{\widetilde{F} - \pi_F^{-1}F}$: $\dim E \leq r-1$, so (2) follows by applying (1) r times.

(3) follows from (2). □

Working in a variety of complete curves $\widetilde{V}$, the number we are after is the number of points of intersections of *complete* subsets of $\widetilde{V}$: counting multiplicities, the number is given by the *degree* of $\widetilde{L}^r \cdot \widetilde{F}$.

Now for the point-conditions. As above, let $\widetilde{V}$ be a variety of complete plane curves of degree d, mapping to $\mathbb{P}^N$ by π, F a locally closed subset of $\mathbb{P}^N$, $r = \dim F$ and $\widetilde{F}$ the proper transform in $\widetilde{V}$ of *the closure* $\overline{F}$ of F. In general, $\widetilde{\ }$ denotes proper transform via π.

Lemma 1. *There exists a point-condition P such that $\overline{P \cap F} = P \cap \overline{F}$, $\widetilde{P \cap \overline{F}} = \widetilde{P} \cap \widetilde{F}$, and* $\dim(\widetilde{P} \cap \widetilde{F}) = r-1$.

Proof. Let π_F be the restriction of π to $\widetilde{F}$. $\overline{P \cap F} = P \cap \overline{F}$ and $\dim(\widetilde{P} \cap \widetilde{F}) = r-1$ are forced by requiring that P cut properly F and $\overline{F} - F$. Next, certainly $\widetilde{P \cap \overline{F}}$ coincides with $\widetilde{P} \cap \widetilde{F}$ outside $\pi_F^{-1}(S)$ for any point-condition P; we have to show that we can choose P so that none of the components of $\widetilde{P} \cap \widetilde{F}$ lies in $\pi_F^{-1}(S)$. Let then $F_i \subset \overline{F}$ be the supports of the components of $\pi_F^{-1}(S)$, and choose the point-condition P so that it cuts properly all the F_i's. Since $\pi_F^{-1}(S)$ itself has dimension (at most) $r-1$, this will force $\dim \pi_F^{-1}(S) \cap \widetilde{P} < r-1$, and we will be done. That a point-condition can be chosen to cut properly any finite choice of subvarieties of $\mathbb{P}^N$ is once more a consequence of the nondegeneracy of the set of point-conditions (Remark 1). □

Following our line of notations, call now *point-conditions in* $\widetilde{V}$ the proper transforms in $\widetilde{V}$ of the point-conditions in $\mathbb{P}^N$; the general ones determine a divisor class $\widetilde{P}$ of $\widetilde{V}$.

From Proposition 1 and Lemma 1, the first part of our basic tool follows:

Theorem I. (1) *Let $\widetilde{V}$ be a variety of complete plane curves of degree d, F an r-dimensional subvariety of $\mathbb{P}^N$ parametrizing a family of reduced curves, and let $\widetilde{F}$ be the proper transform of $\overline{F}$ in $\widetilde{V}$. Then the number of elements (counted with multiplicities) of F containing n_p points and tangent to n_l lines in general position, with $n_p + n_l = r$, is given by $\widetilde{P}^{n_p} \cdot \widetilde{L}^{n_l} \cdot \widetilde{F}$.*

Proof. By repeated applications of Lemma 1, n_p point-conditions $P_1, \dots, P_{n_p}$ can be chosen so that $[\overline{(\bigcap_i P_i) \cap F}]^{\sim} = [(\bigcap_i P_i) \cap \overline{F}]^{\sim} = (\bigcap_i \widetilde{P}_i) \cap \widetilde{F}$. To conclude, it suffices to apply Proposition 1 to $(\bigcap_i P_i) \cap F$. □

The last part of Theorem I concerns intersection multiplicities. It can be proven by induction on n_l; the start and the induction step are consequences of:

Lemma 2. *Let C be an irreducible curve in $\mathbb{P}^N$, such that $C \cap S = \varnothing$ and that the curves in C do not have a common component. Let c be a general point of C; then*

(1) *there exist at most finitely many points $p \in \mathbb{P}^2$ such that $p \in c$ and the point-condition corresponding to p is tangent to C at c;*

(2) *there exist at most finitely many lines $l \subset \mathbb{P}^2$ such that c is properly tangent to l and the line-condition corresponding to l is tangent to C at c.*

Proof. We can assume c is a smooth point of C; if c is reducible as a plane curve, we can in fact assume that all components of c are moving smoothly as c moves on C.

(1) By definition, the point-condition P corresponding to $p \in \mathbb{P}^2$ contains c if and only if $p \in c$. P is tangent to C at c if it contains the tangent line to C at c: let $c' \neq c$ be a point of this line. P contains the line through c and $c' \neq c$ if and only if $p \in c \cap c'$: since all components of c are moving smoothly, this intersection is finite.

(2) Since c and l are properly tangent, then c is a smooth point of the line-condition L_l corresponding to l: therefore, C is tangent to L_l at c if and only if $\psi(C)$ is tangent at $\psi(c)$ to the point-condition corresponding to l (since $C \cap S = \varnothing$, $\psi(c)$ is defined for all $c \in C$). Also, l belongs to a reduced component of $\psi(c)$, since the tangency point is smooth on c. Since c is general in C, we can assume this component is moving smoothly at $\psi(c)$. (2) is then simply the dual of (1): i.e., (1) applied to $\psi(c) \in \psi(C) \subset \mathbb{P}^M$. □

Theorem I. (2) *In the same hypotheses of Theorem* I (1), *the elements containing the given points and properly tangent to the given lines appear with multiplicity* 1.

Proof. We can assume that the curves in F do not have a common component: if they do, factoring it out reduces the statement to the same for lower degree curves.

We will prove that: (a) the statement is true for $n_p = r$, $n_l = 0$; (b) the statement for $n_p = r - k$, $n_l = k$, $k < r$ implies the statement for $n_p = r - k - 1$, $n_l = k + 1$. The assertion will then follow by induction.

(a) It is enough to show that there exists a point-condition P such that $P \cap F$ is reduced, and to apply this fact r times. Now, suppose that is not the case: i.e., suppose that for each point-condition P, $P \cap F$ has some nonreduced component, of dimension $r - 1$. These components would cover a component of F, and the set of point-conditions is 2-dimensional: thus for a general point c in a component of F there would be infinitely many point-conditions tangent to F at c. In particular, they would all be tangent to some curve through c, contradicting Lemma 2 (1).

(b) For $n_p = r - k$, $n_l = k$, the statement says that for general line-conditions $L_1, \dots, L_k$ and point-conditions $P_1, \dots, P_{r-k}$, the intersection $F \cap L_1 \cap \dots \cap L_k \cap P_1 \cap \dots \cap P_{r-k}$ is transversal at all points corresponding to *proper* tangency to the lines. Consequently, the components C_i of the curve $F \cap L_1 \cap \dots \cap L_k \cap P_1 \cap \dots \cap P_{r-k-1}$ that contain these points are reduced and cut transversally by P_{r-k}: to prove the induction step, we must show that there exists a line l in the plane, such that the corresponding line-condition L_{k+1} cuts the C_i transversally at points corresponding to proper tangency to l. By Lemma 2 (2), the set of line-conditions which fail to cut transversally the C_i's at points corresponding to proper tangencies is at most 1-dimensional, in the 2-dimensional set of line-conditions. Therefore, an l as above must exist. □

2 Preliminaries: an intersection formula

In §3 we will construct a 'variety of complete plane cubics' $\widetilde{V}$ by a stack of blow-ups at nonsingular centers over $\mathbb{P}^9$. Corollary I in §1 expresses the characteristic numbers as degrees of intersection of the proper transforms in $\widetilde{V}$ of suitable hypersurfaces of $\mathbb{P}^9$; we introduce here the formula we will use in §4 to compute these intersection degrees.

Let V be a nonsingular variety of dimension n over an arbitrary field, and B a nonsingular closed subvariety of codimension d in V. For $X \hookrightarrow V$ any pure-dimensional subscheme of V, we set

$$B \circ X = c(N_B V) \cap s(B \cap X, X)$$

in the Chow group $A_*(B \cap X)$ of $B \cap X$. We call this the 'full intersection class' of X by B in V.

Lemma. *Denote by* $\{\cdot\}_r$ *the* r*-dimensional component of the class between braces.*

(1) *Let* N *be the pull-back of* $N_B V$ *to* $B \cap X$, $C = C_{B\cap X}X \hookrightarrow N$ *the cone of* $B \cap X$ *in* X, $\mathcal{Q}$ *the universal quotient bundle of rank* d *of* $\mathbb{P}(N \oplus 1)$, p *the projection* $\mathbb{P}(N \oplus 1) \longrightarrow B \cap X$. *Then*

$$B \circ X = p_*(c(\mathcal{Q}) \cap [\mathbb{P}(C \oplus 1)]);$$

(2) $\{B \circ X\}_{\dim X - d} = B \cdot X = j^![X]$;

(3) $\{B \circ X\}_\nu = 0$ *for* $\nu < \dim X - d$, $\nu > \dim B \cap X$.

Proof. (1) Denote by $\mathcal{O}(-1)$ the universal line bundle on $\mathbb{P}(N \oplus 1)$; then $\mathcal{Q} = p^*N \oplus 1/\mathcal{O}(-1)$, and therefore

$$\begin{aligned} p_*(c(\mathcal{Q}) \cap [\mathbb{P}(C \oplus 1)]) &= p_*(c(p^*N) \cap (1 - c_1(\mathcal{O}(1))^{-1} \cap [\mathbb{P}(C \oplus 1)])) \\ &= p_*\left(c(p^*N) \cap \left(\sum_{i\geq 0} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)]\right)\right) \\ &= c(N) \cap s(C) \\ &= g^*c(N_B V) \cap s(B \cap X, X). \end{aligned}$$

(2) See [F, Proposition 6.1(a)] and §6.2 ($j^!$ is the 'Gysin homomorphism').

(3) $\{B \circ X\}_\nu = 0$ for $\nu > \dim B \cap X$ is obvious; $\{B \circ X\}_\nu = 0$ for $\nu < \dim X - d$ follows from (1). □

Let $\widetilde{V}$ be the blow-up of V along B, suppose $X_1, \ldots, X_r$ are pure-dimensional subschemes of V, and let $\widetilde{X}_\nu \subset \widetilde{V}$ be their proper transforms: i.e., the blow-ups of X_ν along $B \cap X_\nu$.

Theorem II. *Suppose that the codimensions of the* X_ν *add to the dimension of* V, *and that the intersection* $\bigcap X_\nu$ *is a proper scheme. With the notation above*

$$\int_{\widetilde{V}} \widetilde{X}_1 \cdot \cdots \cdot \widetilde{X}_r = \int_V X_1 \cdot \cdots \cdot X_r - \int_B \frac{\prod_{\nu=1}^r (B \circ X_\nu)}{c(N_B V)}.$$

Here the first product is taken in $\widetilde{V}$, the second in V and the third in B.

In §4 this formula will be applied to each blow-up in the sequence.

Proof. Let $E = \mathbb{P}(N_B V)$ be the exceptional divisor of the blow-up, and write the maps involved as follows:

$$\begin{array}{ccc} E & \xrightarrow{\ i\ } & \widetilde{V} \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\ j\ } & V \end{array}$$

Theorem II follows from

Claim. *Let* $X_1, \dots, X_r$ *be pure-dimensional subschemes of* V, *and write* $m = \dim V - \operatorname{codim}_V X_1 - \cdots - \operatorname{codim}_V X_r$. *Then*

$$(*)\qquad \pi^*(X_1 \cdot \cdots \cdot X_r) = \widetilde{X}_1 \cdot \cdots \cdot \widetilde{X}_r + i_* \left\{ \frac{\rho^*((B \circ X_1) \cdot \cdots \cdot (B \circ X_r))}{c(N_E \widetilde{V})} \right\}_m$$

in $A_m(\bigcap_i \pi^{-1} X_i)$.

By [F, Proposition 6.7(d)], to prove $(*)$ one must show that the equality holds after (1) pushing it forward on V by π, and (2) pulling it back on E by i. We will show (1) here (which is enough to imply Theorem II), and leave (2) to the interested reader.

By the the projection formula, (1) amounts to

$$X_1 \cdot \cdots \cdot X_r = \pi_*(\widetilde{X}_1 \cdot \cdots \cdot \widetilde{X}_r) + j_* \left\{ \frac{\prod_{\nu=1}^r (B \circ X_\nu)}{c(N_B V)} \right\}_m .$$

Example 12.4.4 in [F] gives

$$X_1 \cdot \cdots \cdot X_r = \pi_*(\widetilde{X}_1 \cdot \cdots \cdot \widetilde{X}_r) + j_* p_*(\mathbb{P}(C_1 \oplus 1) \cdot \cdots \cdot \mathbb{P}(C_r \oplus 1)),$$

where p denotes the projection $\mathbb{P}(N_B V \oplus 1) \longrightarrow B$ and $C_i = C_{B \cap X_i} X_i$ are the normal cones of the imbeddings $B \cap X_i \hookrightarrow X_i$. Thus to prove (1) we may show that

$$(1')\qquad p_*(\mathbb{P}(C_1 \oplus 1) \cdot \cdots \cdot \mathbb{P}(C_r \oplus 1)) = \left\{ \frac{(B \circ X_1) \cdot \cdots \cdot (B \circ X_r)}{c(N_B V)} \right\}_m ,$$

where p is the projection $\mathbb{P}(N \oplus 1) \longrightarrow B$.

To this effect, let $d = \operatorname{codim}_V B$, $\zeta = c_1(\mathscr{O}_{\mathbb{P}(N \oplus 1)}(1))$, and $\mathscr{Q}$ be the universal quotient bundle of rank d over $\mathbb{P}(N \oplus 1)$. Any element A in $A_k(\mathbb{P}(N \oplus 1))$ can be expressed uniquely in the form

$$A = \sum_{\nu=0}^{d} \zeta^\nu \cap p^* \alpha_{k-d+\nu},$$

where $\alpha_j \in A_j(B)$. Setting $\alpha = \bigoplus_{\nu=0}^{d} \alpha_{k-d+\nu} \in A_*(B)$, we say that A *corresponds* to α.

Claim 1. $[\mathbb{P}(C_i \oplus 1)]$ *corresponds to* $B \circ X_i$, $i = 1, \dots, r$.

Indeed, for any $\alpha_j \in A_j B$ and any $\nu \le d$, by Example 3.3.3 in [F]

$$p_*(c(\mathscr{Q}) \cap \zeta^\nu \cap p^* \alpha_j) = \alpha_j;$$

thus the $[\mathbb{P}(C_i \oplus 1)]$ must correspond to $p_*(c(\mathscr{Q}) \cap [\mathbb{P}(C_i \oplus 1)])$. This equals $B \circ X_i$ by (1) of the lemma.

Next, we relate in the above terminology intersections in $\mathbb{P}(N \oplus 1)$ and B. With $n = \dim V = \dim \mathbb{P}(N \oplus 1)$:

Claim 2. *Suppose* $A_i \in A_{k_i}(\mathbb{P}(N \oplus 1))$ *correspond to* $\alpha^{(i)}$, $i = 1, \dots, r$, *and let* $m = k_1 + \cdots + k_r - (r-1)n$. *Then*

$$p_*(A_1 \cdot \cdots \cdot A_r) = \left\{ \frac{\alpha^{(1)} \cdot \cdots \cdot \alpha^{(r)}}{c(N)} \right\}_m .$$

Indeed, by linearity we may assume $A_i = \zeta^{q_i} \cap p^* \alpha^{(i)}$, with $\alpha^{(i)} \in A_{k_i - d + q_i} B$. Setting $q = \sum q_i$ and applying the projection formula, the claim reduces to

$$p_*(\zeta^q \cap p^* \alpha^{(r)}) = s_{q-d}(N) \cap \alpha^{(r)},$$

which amounts to the definition of $s(N) = c(N)^{-1}$ [F, §3.1].

Claims 1 and 2 give $(1')$, concluding the proof of (1). □

We remark that $(*)$ above (and therefore Theorem II) holds for possibly singular V and B, if B is regularly imbedded in V and under conditions that guarantee the intersection products are defined.

One advantage in writing the formula in Theorem II in terms of full intersection classes is that these are often easy to express 'concretely'. In particular:

(i) if $X_\nu = V$, then $B \circ X_\nu = [B]$. Indeed, in this case $s(B \cap X_i, X_i) = s(B, V)$ is the inverse total Chern class $c(N_B V)^{-1}$.

(ii) If X_ν is a divisor then $B \circ X_\nu = e_B X_\nu [B] + j^*[X_\nu]$, where $e_B X$ denotes the multiplicity of X along B and j is the imbedding $B \hookrightarrow V$.

(iii) Similarly, if X_ν has codimension 2 and meets B in an irreducible W, with $\dim W = \dim B - 1$, then $B \circ X_\nu = e_W X_\nu [W] + j^*[X_\nu]$.

(These statement follow from (2), (3) in the lemma.)

By use of Theorem II, the characteristic numbers for a family $F \subset \mathbb{P}^9$ will be expressed in terms of certain full intersection classes related to F (Theorem IV, §4). For F the family of smooth cubics, we will just have to apply (i). To build up Theorem IV, we will need to compute full intersection classes related to point- and line- conditions, using (ii) (see §3); and (iii) will be needed for further computations in §5.

3 A SMOOTH VARIETY OF COMPLETE CUBICS

Assume hereafter that the characteristic of the ground field is $\neq 2, 3$. In this section we will construct a smooth variety of complete plane cubics, by means of a stack of blow-ups over $\mathbb{P}^9$. The same sequence of blow-ups was obtained independently by U. Sterz (cf. [St]); he gives a detailed description in coordinates of each of them, and computes their homology bases and several relations. Our point of view differs from Sterz's in the sense that we need to obtain 'geometric' information regarding the blow-ups, to apply the intersection formula of §2. More precisely, we need for each blow-up $V_{i+1} = \mathrm{B}l_{B_i} V_i$ a description of the intersection ring of each center B_i and the total chern class $c(N_{B_i} V_i)$ of its normal bundle; also, we need the full intersection classes of the proper transforms of point- and line-conditions in each blow-up with respect to the center. The result is

Theorem III. *A smooth variety* $\widetilde{V} = V_5$ *of complete cubics can be obtained by a sequence of* 5 *blow-ups* $V_i = Bl_{B_{i-1}} V_{i-1}$, $i = 1, \dots, 5$, *with* $V_0 = \mathbb{P}^9$, *and where*

(1) $B_0 \cong \mathbb{P}^2$ *is the locus of 'triple lines' in the space* $V_0 = \mathbb{P}^9$ *of plane cubics; the intersection ring of* B_0 *is generated by the hyperplane class* h, *and* $\int h^2 = 1$;

$$c(N_{B_0} V_0) = \frac{(1+3h)^{10}}{(1+h)^3}.$$

(2) B_1 *is a rank-*2 *projective bundle over* B_0;
the intersection ring of B_1 *is generated by the pull-back* h *of* h *and by the class* ε *of the universal line bundle, and*

$$\int_{B_1} h^4 = 0, \quad \int_{B_1} h^3\varepsilon = 0, \quad \int_{B_1} h^2\varepsilon^2 = 1$$

$$\int_{B_1} h\varepsilon^3 = 9, \quad \int_{B_1} \varepsilon^4 = 51;$$

$$c(N_{B_1} V_1) = (1+\varepsilon)\frac{(1+3h-\varepsilon)^{10}}{(1+2h-\varepsilon)^6}.$$

(3) B_2 *is a rank-*3 *projective bundle over* B_1;
the intersection ring of B_2 *is generated by the pull-backs* h, ε *of* h, ε *and by the class* φ *of the universal line bundle, and*

$$\int_{B_2} \varphi^7 = -210, \quad \int_{B_2} \varphi^6 h = -90, \quad \int_{B_2} \varphi^6\varepsilon = -240,$$

$$\int_{B_2} \varphi^5 h^2 = -10, \quad \int_{B_2} \varphi^5 h\varepsilon = 0, \quad \int_{B_2} \varphi^5\varepsilon^2 = 105,$$

$$\int_{B_2} \varphi^4 h^2\varepsilon = 4, \quad \int_{B_2} \varphi^4 h\varepsilon^2 = 18, \quad \int_{B_2} \varphi^4\varepsilon^3 = 42,$$

$$\int_{B_2} \varphi^3 h^2\varepsilon^2 = -1, \quad \int_{B_2} \varphi^3 h\varepsilon^3 = -9, \quad \int_{B_2} \varphi^3\varepsilon^4 = -51,$$

(*all other codimension-*7 *terms have degree* 0);

$$c(N_{B_2} V_2) = (1+\varphi)(1+\varepsilon-\varphi).$$

(4) B_3 *is isomorphic to the blow-up* $Bl_\Delta \mathbb{P}^2 \times \mathbb{P}^2$ *of* $\mathbb{P}^2 \times \mathbb{P}^2$ *along the diagonal; the intersection ring of* B_3 *is generated by the pull-backs* l, m *of the hyperplane classes in the factors of* $\mathbb{P}^2 \times \mathbb{P}^2$, *and by the class* e *of the exceptional divisor, and* $em = el$, $l^3 = m^3 = 0$,

$$\int_{B_3} l^2 m^2 = 1, \quad \int_{B_3} e^2 l^2 = -1,$$

$$\int_{B_3} e^3 l = -3, \quad \int_{B_3} e^4 = -6;$$

$$\begin{aligned} c(N_{B_3}V_3) = 1 &+ 7l + 17m - 16e + 126m^2 + 99lm + 21l^2 \\ &- 315el + 105e^2 + 582lm^2 + 237l^2m - 2517el^2 + 1611e^2l \\ &- 358e^3 + 1026l^2m^2 + 9174e^2l^2 - 3912e^3l + 652e^4 . \end{aligned}$$

(5) *B_4 is isomorphic to B_3;*
the intersection ring of B_4 is therefore generated by l, m, e as above;

$$\begin{aligned} c(N_{B_4}V_4) = 1 - 5l + 5m + 18m^2 - 27lm + 3l^2 + 21el - 7e^2 - 30lm^2 \\ + 75l^2m - 225el^2 + 135e^2l - 30e^3 + 75l^2m^2 . \end{aligned}$$

Also, in these notations:

Full intersection classes. *The full intersection classes with respect to the B_i's of the proper transforms P_i, L_i in V_i of point- and line-conditions are*

$$\begin{array}{ll} B_0 \circ P_0 = 3h , & B_0 \circ L_0 = 2 + 12h , \\ B_1 \circ P_1 = 3h , & B_1 \circ L_1 = 1 + 12h - 2\varepsilon , \\ B_2 \circ P_2 = 3h , & B_2 \circ L_2 = 1 + 12h - 2\varepsilon - \varphi , \\ B_3 \circ P_3 = l + 2m , & B_3 \circ L_3 = 1 + 4l + 8m - 6e , \\ B_4 \circ P_4 = l + 2m , & B_4 \circ L_4 = 1 + l + 5m - 2e . \end{array}$$

The rest of this §3 is devoted to the proof of Theorem III. Most of the notations employed here appear in the following diagram:

$$\begin{array}{ccccccc}
 & & \widetilde{V} = V_5 & & & & \\
 & & \big\downarrow{\scriptstyle \pi_5} & & & & \\
 & & V_4 & \xleftarrow{j_4} & B_4 = \mathbb{P}(\mathscr{L}) & & \\
 & & \big\downarrow{\scriptstyle \pi_4} & & \big\downarrow & & \\
 & & V_3 & \xleftarrow{j_3} & B_3 = S_3 & \underset{\sim}{\xleftarrow{\phi_3}} & Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\
 & & \big\downarrow{\scriptstyle \pi_3} & & \big\downarrow & & \| \\
B_2 & \xrightarrow{j_2} & V_2 & \longleftarrow & S_2 & \underset{\sim}{\xleftarrow{\phi_2}} & Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\
{\scriptstyle \mathbb{P}^3\text{-bundle}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow & & \| \\
B_1 & \xrightarrow{j_1} & V_1 & \longleftarrow & S_1 & \underset{\sim}{\xleftarrow{\phi_1}} & Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\
{\scriptstyle \mathbb{P}^2\text{-bundle}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow & & \big\downarrow \\
v_3(\check{\mathbb{P}}^2) = B_0 & \xrightarrow{j_0} & \mathbb{P}^9 = V_0 & \longleftarrow & S_0 & \xleftarrow{\phi_0} & \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2
\end{array}$$

Here S_0 is the locus of nonreduced cubics, and $B_0 = v_3(\check{\mathbb{P}}^2) \hookrightarrow \mathbb{P}^9$ is the Veronese of triple lines. Each B_i is the center of the blow-up $V_{i+1} = \mathrm{B}l_{B_i} V_i$; S_{i+1} denotes the proper transform of S_i under this blow-up.

Also, $\mathscr{L}$ is a certain subline bundle of the normal bundle $N_{B_3}V_3$ of B_3 in V_3. Δ is the diagonal in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

Finally, P_0, L_0 will respectively be point- and line-conditions in $\mathbb{P}^9$; P_i, L_i will be the proper transforms of P_{i-1}, L_{i-1}, and E_i will be the exceptional divisor of the ith blow-up. The P_i's and L_i's will be called 'point-conditions' and 'line-conditions' *in* V_i. We will also say that point- and line-conditions are 'in general position' if the corresponding points and lines are.

§§3.0–3.5 describe the sequence of blow-ups in some detail. The aim is to 'separate' the line-conditions by blowing-up nonsingular subvarieties; thus, we will generally choose as centers the 'biggest' nonsingular components of the intersection of the line-conditions. In $\mathbb{P}^9$ the intersection of the line-condition is supported on the 4-dimensional variety S_0 parametrizing non-reduced curves. This is the image of a bijective map $\phi_0 \colon \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \longrightarrow \mathbb{P}^9$; ϕ_0 ramifies along the diagonal, mapping to the 2-dimensional locus B_0 of 'triple lines'. This latter is our choice for the first blow-up.

Each of §§3.0–3.5 is organized as follows: we find on each V_i the intersection of all line-conditions, and we choose a center B_i for the next blow-up (a nonsingular subvariety or component of the intersection of the line-conditions); then we obtain the information collected in Theorem III. In particular, we describe the intersection rings of the B_i's, and we compute the total Chern class $c(N_{B_i}V_i)$ of the normal bundle to B_i in V_i. Next, we examine the geometry of the situation in more detail, to obtain the information we will need in the following stages. Also, we compute the multiplicities $e_{B_i}P_i$, $e_{B_i}L_i$ of the conditions along the centers (in fact, $e_{B_i}P_i$ will always be 0), and the pull-backs $j_i^*P_i$, $j_i^*L_i$: this is the information contained in the full intersection classes of the conditions with respect to the centers.

To prove that V_5 is a 'variety of complete cubics' amounts to proving that the intersection of its line-conditions is empty; this will be shown in §3.5. Equivalently, one can show that V_5 dominates the graph of the rational map ψ of §1; a proof in these terms can be found in [St, II, §4].

3.0 The $\mathbb{P}^9$ of plane cubics

Let Q be a 3-dimensional vector space over an algebraically closed field of characteristic $\neq 2, 3$, and consider $\mathbb{P}^9 = \mathbb{P}(\mathrm{Sym}^3 \check{Q})$, the projective space parametrizing cubic curves in the plane $\mathbb{P}^2 = \mathbb{P}Q$. x_0, x_1, x_2 (resp., $a_0, \ldots, a_9$) will be homogeneous coordinates in $\mathbb{P}^2$ (resp., in $\mathbb{P}^9$): the point $(a_0 \colon \ldots \colon a_9) \in$

$\mathbb{P}^9$ is associated with the cubic of equation

$$a_0x_0^3 + a_1x_0^2x_1 + a_2x_0^2x_2 + a_3x_0x_1^2 + a_4x_0x_1x_2 + a_5x_0x_2^2 + a_6x_1^3 + a_7x_1^2x_2 + a_8x_1x_2^2 + a_9x_2^3 = 0.$$

We will write K simultaneously for the cubic K in $\mathbb{P}^2$, a cubic polynomial giving K in terms of the coordinates $(x_0 : x_1 : x_2)$, and the corresponding point $K \in \mathbb{P}^9$. Similarly, $\lambda \in \check{\mathbb{P}}^2$ will stand for both the line λ in $\mathbb{P}^2$ and a corresponding linear function in terms of $(x_0 : x_1 : x_2)$.

We observed already (see §1) that the point-conditions P_0 in $\mathbb{P}^9$ are hyperplanes, while the line-conditions L_0 form hypersurfaces of degree 4. Explicitly, the line-condition corresponding to the line $x_0 = 0$ is the discriminant of the polynomial in x_1, x_2

$$a_6x_1^3 + a_7x_1^2x_2 + a_8x_1x_2^2 + a_9x_2^3,$$

hence has equation

$$(*) \qquad a_7^2a_8^2 + 18a_6a_7a_8a_9 - 4a_6a_8^3 - 4a_7^3a_9 - 27a_6^2a_9^2 = 0.$$

The following facts about line-conditions are independent of the corresponding line, therefore can be checked on $(*)$:

Lemma 0.1. *Let L be the line-condition in $\mathbb{P}^9$ corresponding to $\lambda \in \check{\mathbb{P}}^2$. Then:*

(1) *If $K \in L$, then L is smooth at K if and only if K intersects λ with multiplicity exactly 2 at a point. In particular, the line-conditions are generically smooth along the locus S_0 of nonreduced cubics.*
(2) *If K intersects λ with multiplicity 3 at a point, then L has multiplicity 2 at K. In particular, the line-conditions have multiplicity 2 along the locus B_0 of triple lines.*
(3) *The tangent hyperplane to L at a smooth point K consists of the cubics containing the point of tangency of K to λ. The tangent cone in $V_0 = \mathbb{P}^9$ to L at a cubic K intersecting λ in a triple point p is supported on the hyperplane in V_0 consisting of the cubics containing p.*

The intersection of all line-conditions consists of the locus of non-reduced cubics, which we denote S_0. S_0 is the image of the 1-1 morphism

$$\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \xrightarrow{\phi_0} \mathbb{P}^9$$
$$(\lambda, \mu) \mapsto \lambda\mu^2$$

which maps the pair of lines with equations $\{\lambda = 0\}$, $\{\mu = 0\}$ to the cubic of equation $\{\lambda\mu^2 = 0\}$. If Δ is the diagonal in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$, $\phi_0(\Delta)$ is the locus B_0 of triple lines.

Lemma 0.2. *The restriction of* $\phi_0 \colon \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 - \Delta \longrightarrow S_0 - B_0$ *is an isomorphism. In particular,* $S_0 - B_0$ *is nonsingular.*

Proof. The locus where $d\phi_0$ is an isomorphism is clearly invariant under projective transformations of $\mathbb{P}^2$, and the group of projective transformations acts transitively on $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 - \Delta$. □

In fact, S_0 is singular along B_0 (see Remarks 1.4); we choose B_0 as the center of the first blow-up.

Lemma 0.3. B_0 *is the third Veronese imbedding of* $\check{\mathbb{P}}^2$ *in* $\mathbb{P}^9$, *thus a nonsingular 2-dimensional subvariety of* V_0. *The tangent space to* B_0 *in* V_0 *at a point* $\lambda^3 \in B_0$ *consists of the cubics vanishing twice along* λ.

Proof. ϕ_0 restricts on $\check{\mathbb{P}}^2 \cong \Delta \longrightarrow B_0$ to $\lambda \mapsto \lambda^3$, the 3^{rd} Veronese imbedding v_3. The last assertion is checked by differentiating $\lambda \mapsto \lambda^3$. □

We can get now the information needed for Theorem III (1):

Theorem III (1). (i) *The intersection ring of* $B_0 \cong \mathbb{P}^2$ *is generated by the hyperplane class* h;

(ii) $c(N_{B_0}V_0) = (1+3h)^{10}/(1+h)^3$.

Proof. (i) is just setting the notation;

(ii) the class of the hyperplane in $V_0 \cong \mathbb{P}^9$ pulls-back to $3h$ on B_0, and $c(N_{B_0}) = j_0^* c(T\mathbb{P}^9)/c(TB_0)$. □

We also find

Lemma 0.4. $j_0^* P_0 = 3h$, $j_0^* L_0 = 12h$; *the full intersection classes of a point-conditions and line-conditions with respect to* B_0 *are*

$$B_0 \circ P_0 = 3h, \qquad B_0 \circ L_0 = 2 + 12h.$$

Proof. The pull-back of the hyperplane class from V_0 to B_0 is $3h$. B_0 is not contained in any point-condition, and the line-conditions have multiplicity 2 along B_0 by 0.1. □

3.1 The first blow-up. Let $V_1 = \mathrm{B}l_{B_0}V_0$, write $\pi_1 \colon V_1 \longrightarrow V_0$ for the blow-up map, E_1 for the exceptional divisor, and denote by S_1, P_1, L_1 the proper transforms of S_0, P_0, L_0. Then $P_1 = \pi_1^* P_0$, $L_1 = \pi_1^* L_0 - 2E_1$ as divisor classes.

We will see here that the line-conditions in V_1 intersect along the smooth 4-dimensional proper transform S_1 of S_0 and along a smooth 4-dimensional subvariety B_1 of the exceptional divisor E_1 (Proposition 1.2). We will choose B_1 as the center for the second blow-up.

We determine now the intersection of the line-conditions in V_1. Since $V_1 - E_1 \cong V_0 - B_0$, S_1 must be a component of the intersection. To find components contained in E_1, identify E_1 with the projective bundle $\mathbb{P}(N_{B_0}V_0)$

over B_0; the key observation is

Lemma 1.1. *There is an imbedding* $N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5 \hookrightarrow N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9$ *of vector bundles over* $\check{\mathbb{P}}^2$.

Proof. We have the exact sequences on $B_0 \cong \check{\mathbb{P}}^2 = \mathbb{P}(\check{Q})$

$$0 \longrightarrow \mathcal{O}_{\check{\mathbb{P}}^2} \longrightarrow \check{Q} \otimes \mathcal{O}_{\check{\mathbb{P}}^2}(1) \longrightarrow T\check{\mathbb{P}}^2 \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_{\check{\mathbb{P}}^2} \longrightarrow \operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}_{\check{\mathbb{P}}^2}(2) \longrightarrow T\mathbb{P}^5 \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_{\check{\mathbb{P}}^2} \longrightarrow \operatorname{Sym}^3 \check{Q} \otimes \mathcal{O}_{\check{\mathbb{P}}^2}(3) \longrightarrow T\mathbb{P}^9 \longrightarrow 0:$$

the first is the standard Euler sequence on $\mathbb{P}\check{Q}$; the second and third are the pull-backs of the Euler sequences on $\mathbb{P}(\operatorname{Sym}^2 \check{Q})$ and $\mathbb{P}(\operatorname{Sym}^3 \check{Q})$ via the Veronese imbeddings $v_2\colon \check{\mathbb{P}}^2 \longrightarrow \mathbb{P}^5$ and $v_3\colon \check{\mathbb{P}}^2 \longrightarrow \mathbb{P}^9$. From these we get

$$N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5 = \frac{\operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(2)}{\check{Q} \otimes \mathcal{O}(1)}, \qquad N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9 = \frac{\operatorname{Sym}^3 \check{Q} \otimes \mathcal{O}(3)}{\check{Q} \otimes \mathcal{O}(1)}.$$

The claimed imbedding $N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5 \hookrightarrow N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9$ is induced by the map

$$\operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(-1) \longrightarrow \operatorname{Sym}^2 \check{Q} \otimes \check{Q} \longrightarrow \operatorname{Sym}^3 \check{Q}:$$

this gives an imbedding

$$\operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(2) = \operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(-1) \otimes \mathcal{O}(3) \hookrightarrow \operatorname{Sym}^3 \check{Q} \otimes \mathcal{O}(3),$$

that induces an imbedding on the quotients. □

We can then define a 4-dimensional smooth subvariety B_1 of E_1, i.e.

$$B_1 = \mathbb{P}(N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5) \hookrightarrow \mathbb{P}(N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9) = E_1.$$

Notice that the fiber of $\operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(-1) \hookrightarrow \operatorname{Sym}^3 \check{Q}$ over $\lambda \in \mathbb{P}(\check{Q})$ consists of the cubic polynomials over Q divisible by λ. The fiber of $B_1 = \mathbb{P}(N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5)$ is then $\mathbb{P}(TR_\lambda/TB_0)$,where $R_\lambda \cong \mathbb{P}^5$ is the subspace of V_0 consisting of the cubics vanishing along (i.e. containing) the line λ. Also recall (Lemma 0.3) that the tangent space to B_0 in $\mathbb{P}^9$ consists of the cubics vanishing *twice* along λ. The information carried by a point of B_1 consists then of a line λ and of the web of conics with given proper intersection with λ: i.e., of λ and of two points on λ. Of course these are the 'complete conics' supported on a double line: $\mathbb{P}(N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5)$ is the exceptional divisor in the space of complete conics (cf. [CX, 2.2]). We will refer to points of B_1 as to 'lines with distinguished pairs of points'.

Proposition 1.2. *The set-intersection of all line-conditions in* V_1 *is contained in the union of the smooth* 4*-folds* S_1 *and* B_1.

Proof. B_1 is a projective bundle over $\mathbb{P}^2$, therefore it is smooth; the smoothness of S_1 is proved in Lemma 1.3 below.

We have to show that the line-conditions intersect along B_1 over B_0, and this can be checked fiberwise. As observed above, the fiber of B_1 over a triple line $\mu^3 \in B_0$ is $\mathbb{P}(T_{\mu^3}R_\mu/T_{\mu^3}B_0)$, where $R_\mu \cong \mathbb{P}^5$ is the subspace of V_0 consisting of the cubics containing μ; on the other hand, by Lemma 0.1 the intersection of the tangent cones to the line-conditions (in $\mathbb{P}^9$) at $\mu^3 \in B_0$ is precisely R_μ, so the assertion follows. □

B_1 will be the center for the next blow-up.

Theorem III (2). B_1 *is a* $\mathbb{P}^2$*-bundle over* B_0.

(i) *The intersection ring of* B_1 *is generated by the pull-back* h *of* h *via* $B_1 \longrightarrow B_0$ *and the pull-back* ε *of* E_1 *via* $j_1 \colon B_1 \hookrightarrow V_1$, *and*

$$\int_{B_1} h^4 = 0, \quad \int_{B_1} h^3\varepsilon = 0, \quad \int_{B_1} h^2\varepsilon^2 = 1, \quad \int_{B_1} h\varepsilon^3 = 9, \quad \int_{B_1} \varepsilon^4 = 51$$

(ii) $c(N_{B_1}V_1) = (1+\varepsilon)(1+3h-\varepsilon)^{10}/(1+2h-\varepsilon)^6$.

Proof. (i) The universal line bundle on $\mathbb{P}(N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5) = B_1$ is the restriction of the one on $\mathbb{P}(N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9) = E_1$, i.e. ε (Lemma 1.1). The first assertion follows then from [F, Example 8.3.4]. Moreover, via $B_1 \longrightarrow B_0$, $1-\varepsilon+\varepsilon^2-\varepsilon^3+\varepsilon^4$ pushes forward to

$$s(N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5) = \frac{(1+h)^3}{(1+2h)^6} = 1 - 9h + 51h^2$$

by [F, Chapter 4], so that the relations follow from the projection formula.

(ii) The normal bundle to B_1 in V_1 is an extension of $N_{B_1}E_1$ and $N_{E_1}V_1$. By Lemma 1.1, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_{B_1} & \longrightarrow & N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5 \otimes \mathscr{O}(1) & \longrightarrow & T_{B_1|B_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}_{E_1} & \longrightarrow & N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9 \otimes \mathscr{O}(1) & \longrightarrow & T_{E_1|B_0} & \longrightarrow & 0 \end{array},$$

so that

$$c(N_{B_1}E_1) = \frac{c(N_{v_3(\check{\mathbb{P}}^2)}\mathbb{P}^9 \otimes \mathscr{O}(1))}{c(N_{v_2(\check{\mathbb{P}}^2)}\mathbb{P}^5 \otimes \mathscr{O}(1))} = \frac{(1+3h-\varepsilon)^{10}}{(1+2h-\varepsilon)^6}.$$

On the other hand, $c(N_{E_1}V_1) = 1+\varepsilon$; thus (ii) follows from the Whitney product formula. □

We proceed next to a closer analysis of the varieties involved at this stage.

Consider the triple line $x_0^3 \in B_0 \hookrightarrow V_0 = \mathbb{P}^9$. Setting $a_0 = 1$, affine coordinates for V_0 at x_0^3 are $(a_1, \dots, a_9)$, and

$$\begin{aligned} 3a_3 - a_1^2 &= 0, & 3a_4 - 2a_1a_2 &= 0, & 3a_5 - a_2^2 &= 0, \\ 9a_6 - a_1a_3 &= 0, & 3a_7 - a_2a_3 &= 0, & 3a_8 - a_1a_5 &= 0, \\ 9a_9 - a_2a_5 &= 0, \end{aligned}$$

are equations for B_0 in a neighborhood of x_0^3. Thus we can choose coordinates $(b_1, \dots, b_9)$ in an open in $V_1 = \mathrm{B}l_{B_0}V_0$ so that

$$\begin{aligned} b_1 &= a_1, & b_2 &= a_2, & b_3 &= 3a_3 - a_1^2, \\ b_4b_3 &= 3a_4 - 2a_1a_2, & b_5b_3 &= 3a_5 - a_2^2, & b_6b_3 &= 9a_6 - a_1a_3, \\ b_7b_3 &= 3a_7 - a_2a_3, & b_8b_3 &= 3a_8 - a_1a_5, & b_9b_3 &= 9a_9 - a_2a_5. \end{aligned}$$

With this choice, $b_3 = 0$ is the equation for the exceptional divisor E_1, and $(b_4, \dots, b_9)$ are coordinates for the fiber of E_1 over a point of B_0.

Recall (§3.0) that S_0 is the image of a 1-1 morphism $\phi_0: \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \to \mathbb{P}^9$, which restricts to an isomorphism on $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 - \Delta$, where Δ is the diagonal in $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

Lemma 1.3. *The proper transform S_1 of S_0 is nonsingular, in fact isomorphic to $Bl_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.*

Proof. We will show that ϕ_0 lifts to an isomorphism $\phi_1: \mathrm{B}l_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \to S_1$ compatible with ϕ_0; that is, such that the following diagram commutes:

$$(*) \qquad \begin{array}{ccc} \mathrm{B}l_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 & \xrightarrow[\sim]{\phi_1} & S_1 \hookrightarrow V_1 \\ \downarrow & & \downarrow \\ \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 & \xrightarrow{\phi_0} & S_0 \hookrightarrow V_0 \end{array} .$$

Let e be the exceptional divisor of $\mathrm{B}l_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$. A morphism ϕ_1 exists by the universal property of blow-ups, and restricts to an isomorphism on $\mathrm{B}l_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 - e$ by Lemma 0.2 (2); so we only need to check that $(d\phi_1)_p$ is injective for $p \in e$. This matter is local and invariant under projective transformations of $\mathbb{P}^2$, so we can assume p is in the fiber of $(x_0, x_0) \in \Delta$. Choose local coordinates $(\alpha_1, \alpha_2; u_1, u_2)$ at (x_0, x_0) so that $(\alpha_1, \alpha_2; u_1, u_2)$ corresponds to

$$(x_0 + (\alpha_1 + u_1)x_1 + (\alpha_2 + u_2)x_2, x_0 + \alpha_1x_1 + \alpha_2x_2) \in \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2.$$

Equations for Δ are then $u_1 = 0$, $u_2 = 0$. Therefore, we can choose coordinates $(\alpha_1, \alpha_2; u, t)$ in an open set (that we can assume contains p) in $\mathrm{B}l_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ so that

$$\alpha_1 = a_1, \qquad \alpha_2 = a_2, \qquad u = u_1, \qquad ut = u_2;$$

the equation for e is $u = 0$.

By the commutativity of $(*)$, in terms of the coordinates given for V_1 and these coordinates for $\mathrm{B}l_\Delta\check{\mathbb{P}}^2\times\check{\mathbb{P}}^2$, ϕ_1 is given explicitly by

$$(**)\quad (\alpha_1,\alpha_2;u,t)\mapsto(3\alpha_1+u,3\alpha_2+ut,-u^2,2t,t^2,2\alpha_1,2\alpha_1 t,2\alpha_2 t,2\alpha_2 t^2),$$

with nondegenerate jacobian, as needed. □

Remarks 1.4.

(1) We will let e denote the exceptional divisor in $\mathrm{B}l_\Delta\check{\mathbb{P}}^2\times\check{\mathbb{P}}^2$ (and its divisor class).

(2) $\phi_1^* E_1 = 2e$: S_1 is tangent to E_1 along e. Consequently, S_0 has multiplicity 2 along B_0: it is indeed singular along it.

(3) A point in e can be visualized as a 'double line with distinguished point'. As a pair of lines $(\lambda,\mu)\in\check{\mathbb{P}}^2\times\check{\mathbb{P}}^2$ approaches an element $(\nu,\nu)\in\Delta$ along some curve, their intersection $\lambda\cap\mu$ approaches a specific point on ν. Elements in e record this information. If $(\alpha_1,\alpha_2,u,t)\notin e$ (i.e. if $u\neq 0$), then the corresponding pair of lines intersects in the point $(\alpha_1 t-\alpha_2:\ -t:\ 1)$; if $(\alpha_1,\alpha_2,0,t)\in e$, then $(\alpha_1 t-\alpha_2:\ -t:\ 1)$ are the coordinates of the 'distinguished point' on the line $x_0+\alpha_1 x_1+\alpha_2 x_2=0$.

Lemma 1.5. (1) *B_1 intersects S_1 along e.*

(2) *The line conditions in V_1 are generically smooth and tangent to E_1 along B_1.*

Proof. (1) This is easily checked using the explicit expression $(**)$ for ϕ_1 in the proof of Lemma 1.3. By invariance under projective transformations, we can assume $\lambda=x_0$; equations for R_λ are then $a_6=a_7=a_8=a_9=0$, and the check is immediate.

(2) By the invariance under projective transformations, it is enough to verify the claim for the line-condition corresponding to $x_2=0$, and we can restrict to the open set on which our local coordinates for V_1 hold. In terms of these coordinates, the proper transform to the line-condition has equation

$$4b_3+(3b_6-2b_1)^2=0,$$

and the assertion is easily checked. □

Lemma 1.6. $j_1^* P_1=3h$, $j_1^* L_1=12h-2\varepsilon$. *The full intersection classes of point- and line-conditions with respect to B_1 are*

$$B_1\circ P_1=3h,\qquad B_1\circ L_1=1+12h-2\varepsilon.$$

Proof. $P_1=\pi_1^* P_0$, $L_1=\pi_1^* L_0-2E_1$; P_1 does not contain B_1, and $e_{B_1}L_1=1$ follows from Lemma 1.5 (2). □

Remarks 1.7.

(1) In terms of the descriptions of e and B_1, ϕ_1 acts on e by mapping the line λ with the distinguished point p into the triple line λ with the

distinguished *double* point p. Therefore, ϕ_1 maps the fiber of e over λ to a nonsingular conic in the fiber of B_1 over λ^3.

(2) Using the last remark and $(**)$ in the proof of 1.3, one gets equations for B_1 in terms of the local coordinates in V_1:

$$b_3 = 0, \qquad 3b_6 - 2b_1 = 0, \qquad 3b_7 - b_1 b_4 = 0,$$
$$3b_8 - b_2 b_4 = 0, \qquad 3b_9 - 2b_2 b_5 = 0.$$

(3) Lemma 1.5 (2) can be stated more precisely:

Consider a point $\tilde{\lambda} \in B_1$, i.e. a line λ with distinguished points p_1, p_2.

Then the line-condition in V_1 corresponding to a line μ is nonsingular at $\tilde{\lambda}$ if $p_1 \notin \mu$ and $p_2 \notin \mu$.

The check is again immediate, for the equation of the proper transform of the line-condition corresponding to $x_2 = 0$.

3.2. The second blow-up. Let $V_2 = Bl_{B_1} V_1$, write $\pi_2 : V_2 \longrightarrow V_1$ for the blow-up map, E_2 for the exceptional divisor, and denote by $\widetilde{E}_1$, S_2, P_2, L_2 the proper transforms of E_1, S_1, P_1, L_1. Then $P_2 = \pi_2^* P_1$, and $L_2 = \pi_2^* L_1 - E_2$ (Lemma 1.5 (2)).

In V_2, we will see that the line-conditions intersect in the proper transform S_2 of S_1 and in a smooth 7-dimensional subvariety B_2 of the exceptional divisor E_2 (Proposition 2.1). B_2 will be the new center of blow-up.

Set $B_2 = \widetilde{E}_1 \cap E_2$. B_2 is the exceptional divisor of the blow-up of E_1 at B_1, therefore (see §3.1)

$$B_2 = \mathbb{P}(N_{B_1} E_1) = \mathbb{P}\left(\frac{\operatorname{Sym}^3 \check{Q} \otimes \mathcal{O}(3)}{\operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(2)} \otimes \mathcal{O}_{B_1}(1)\right)$$

is a $\mathbb{P}^3$-bundle over B_1. In particular, B_2 is *smooth.*

Proposition 2.1. *The set-intersection of all line-conditions in V_2 is contained in the union of S_2 and the 7-dimensional smooth variety $B_2 = \widetilde{E}_1 \cap E_2$.*

Proof. $V_2 - E_2 \cong V_1 - B_1$, thus S_2 is a component of the intersection. By Lemma 1.5 (2), the line-conditions in V_1 are generically tangent to E_1, so their proper transforms all intersect E_2 along $\widetilde{E}_1 \cap E_2 = B_2$. □

The center for the next blow-up will be B_2.

Theorem III (3). *B_2 is a $\mathbb{P}^3$-bundle on B_1.*

(i) *The intersection ring of B_2 is generated by the pull-backs h, ε of h, ε via the projection $B_2 \longrightarrow B_1$, and the pull-back $\varphi = j_2^* E_2$ of E_2 via*

$j_2 \colon B_2 \longrightarrow V_2$. *Also,*

$$\int_{B_2} \varphi^7 = -210, \quad \int_{B_2} \varphi^6 h = -90, \quad \int_{B_2} \varphi^6 \varepsilon = -240,$$
$$\int_{B_2} \varphi^5 h^2 = -10, \quad \int_{B_2} \varphi^5 h\varepsilon = 0, \quad \int_{B_2} \varphi^5 \varepsilon^2 = 105,$$
$$\int_{B_2} \varphi^4 h^2 \varepsilon = 4, \quad \int_{B_2} \varphi^4 h\varepsilon^2 = 18, \quad \int_{B_2} \varphi^4 \varepsilon^3 = 42,$$
$$\int_{B_2} \varphi^3 h^2 \varepsilon^2 = -1, \quad \int_{B_2} \varphi^3 h\varepsilon^3 = -9, \quad \int_{B_2} \varphi^3 \varepsilon^4 = -51,$$

hold (*all other codimension*-7 *terms have degree* 0).

(ii) $c(N_{B_2} V_2) = (1+\varphi)(1+\varepsilon-\varphi)$.

Proof. (i) $B_2 = \mathbb{P}(N_{B_1} E_1)$, with universal line bundle induced from $E_2 = \mathbb{P}(N_{B_1} V_1)$, so the first assertion follows. Moreover, $1-\varphi+\varphi^2-\varphi^3+\varphi^4-\varphi^5+\varphi^6-\varphi^7$ pushes forward to

$$s(N_{B_1} E_1) = \frac{(1+2h-\varepsilon)^6}{(1+3h-\varepsilon)^{10}},$$

and the relations follow directly by Lemma 1.7 and the projection formula.

(ii) $B_2 = E_2 \cap \widetilde{E}_1$, so that $c(N_{B_2} V_2) = c(N_{E_2} V_2) c(N_{\widetilde{E}_1} V_2)$. □

We now obtain a more detailed description of the situation, for future reference. As for S_2:

Lemma 2.1. S_2 *is isomorphic to* S_1, *hence to* $Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

Proof. By Lemma 1.5 (1), S_2 is the blow-up of S_1 along a divisor, thus it is isomorphic to S_1. □

A description of B_2 is gotten as follows:

Let $\tilde{\lambda}$ be a point in B_1 mapping to $\lambda^3 \in B_0$ via $B_1 \longrightarrow B_0$ (i.e. 'λ with two distinguished points'). The fiber of B_2 above $\tilde{\lambda}$ can be identified with the space $\mathbb{P}((\mathrm{Sym}^3 \check{Q})_{\tilde{\lambda}}/(\mathrm{Sym}^2 \check{Q})_{\tilde{\lambda}})$, where $(\mathrm{Sym}^2 \check{Q})_{\tilde{\lambda}} \hookrightarrow (\mathrm{Sym}^3 \check{Q})_{\tilde{\lambda}}$ is the multiplication by λ. $\mathbb{P}((\mathrm{Sym}^3 \check{Q})_{\tilde{\lambda}}/(\mathrm{Sym}^2 \check{Q})_{\tilde{\lambda}})$ is the 3-dimensional space of cubics on λ: a point in the fiber of B_2 above $\tilde{\lambda}$ corresponds then to a triple of points on λ, and we will refer to points of B_2 as to lines with a pair of distinguished points *and* a triple of distinguished points.

Lemma 2.2. (1) B_2 *intersects* $S_2 \cong Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ *along* e.

(2) *The line-conditions in* V_2 *are generically smooth along* B_2.

Proof. (1) Recall that S_1 is tangent to E_1 along e (Remark 1.4 (2)). Thus $S_2 \cap E_2 \subset \widetilde{E}_1 \cap E_2 = B_2$.

(2) By Lemma 1.5 (2), the line-conditions in V_1 are generically smooth along B_1. □

This gives us the additional information about conditions we will need in the computation in §4:

Lemma 2.3. $j_2^*P_2 = 3h$, $j_2^*L_2 = 12h - 2\varepsilon - \varphi$. *The full intersection classes for point- and line-conditions with respect to* B_2 *are*

$$B_2 \circ P_2 = 3h \quad , \quad B_2 \circ L_2 = 1 + 12h - 2\varepsilon - \varphi .$$

Proof. $P_2 = \pi_2^* P_1$; $L_2 = \pi_2^* L_1 - E_2$. P_2 does not contain B_2, and $e_{B_2} L_2 = 1$ follows from Lemma 2.2 (2). □

In the local coordinates given for V_1 in §3.1, equations for B_1 are

$$\begin{aligned} b_3 = 0, \quad 3b_6 - 2b_1 = 0, \quad & 3b_7 - b_1 b_4 = 0, \\ 3b_8 - b_2 b_4 = 0, \quad & 3b_9 - 2b_2 b_5 = 0, \end{aligned}$$

(Remark 1.7 (2)) thus we can choose coordinates $(c_1, \dots, c_9)$ in an open set in V_2 so that

$$\begin{array}{lll} c_1 = b_1, & c_2 = b_2, & c_3 c_6 = b_3, \\ c_4 = b_4, & c_5 = b_5, & c_6 = 3b_6 - 2b_1, \\ c_7 c_6 = 3b_7 - b_1 b_4, & c_8 c_6 = 3b_8 - b_2 b_4, & c_9 c_6 = 3b_9 - 2b_2 b_5. \end{array}$$

In the coordinates $(c_1, \dots, c_9)$, equations for E_2 and $\widetilde{E}_1$ are $c_6 = 0$ and $c_3 = 0$ respectively.

Recall that S_1 is the isomorphic image of a map $\phi_1 \colon Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \hookrightarrow V_1$ (see Lemma 1.3) given in local coordinates by

$$(\alpha_1, \alpha_2; u, t) \mapsto (3\alpha_1 + u, 3\alpha_2 + ut, -u^2, 2t, t^2, 2\alpha_1, 2\alpha_1 t, 2\alpha_2 t, 2\alpha_2 t^2).$$

Now ϕ_1 lifts to a map $\phi_2 \colon Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \hookrightarrow V_2$; a local coordinate expression for ϕ_2 is

$$(*) \qquad (\alpha_1, \alpha_2; u, t) \mapsto (3\alpha_1 + u, 3\alpha_2 + ut, \frac{u}{2}, 2t, t^2, -2u, t, t^2, t^3).$$

Remark 2.4.

(1) From this it follows $\phi_2^* E_2 = e$.

(2) Using $(*)$, one checks that in terms of the descriptions of e as set of lines with distinguished point, and of B_2 as set of lines with distinguished pair and triple of points, ϕ_2 acts $e \longrightarrow B_2$ by mapping the line λ with distinguished point p to the line λ with distinguished double point p and triple point p.

Lemma 2.5. *Let* $\bar{\lambda}$ *be a point on* B_2, *i.e. a line* λ *with distinguished pair of points* p_1, p_2 *and triple of points* q_1, q_2, q_3, *and consider the line-condition* L_μ *in* V_2 *corresponding to a line* $\mu \neq \lambda$. *Then*:

(1) L_μ *is tangent to* E_2 *at* $\bar{\lambda}$ *if* $\exists i, p_i \in \mu$;

(2) L_μ *is tangent to* $\widetilde{E}_1$ *at* $\bar{\lambda}$ *if* $\exists i, q_i \in \mu$.

Proof. We can assume $\lambda = x_0$, $\mu = x_1$, by invariance under projective transformations. In local coordinates, the equation for L_μ is then

$$4c_3c_5^3 + c_6c_9^2 = 0,$$

and coordinates for $\bar{\lambda}$ have $c_5 = 0$ if the pair touches μ, $c_9 = 0$ if the triple touches μ. The verifications are immediate. □

3.3. The third blow-up. Let $V_3 = \mathrm{B}l_{B_2}V_2$, write $\pi_3\colon V_3 \longrightarrow V_2$ for the blow-up map, E_3 for the exceptional divisor, and denote by S_3, P_3, L_3 the proper transforms of S_2, P_2, L_2. Then $P_3 = \pi_3^*P_2$, and $L_3 = \pi_3^*L_2 - E_3$ (Lemma 2.2 (2)).

In V_3 the line-conditions will intersect in the proper transform S_3 of S_2, a 4-dimensional smooth variety isomorphic to the blow-up of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the diagonal (Lemma 3.1, Proposition 3.2). We will choose S_3 as the center B_3 for the fourth blow-up.

We first of all remark:

Lemma 3.1. *S_3 is isomorphic to S_2, hence to $Bl_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.*

Proof. By Lemma 2.2 (1), S_3 is the blow-up of S_2 along a divisor. □

Then

Proposition 3.2. *The intersection of all line-conditions in V_3 is supported on the 4-dimensional smooth irreducible variety S_3.*

Proof. We have to verify that the line-conditions intersect in E_3 only along $S_3 \cap E_3$. Since B_2 has codimension 2 in V_2, E_3 is a $\mathbb{P}^1$-bundle over B_2. A general line-condition is smooth at $\bar{\lambda} \in B_2$ (Lemma 2.2 (2)), thus the line-conditions in V_3 can intersect in at most one point over each $\bar{\lambda} \in B_2$. We have then to check that the line-conditions in V_3 can intersect in E_3 only above $B_2 \cap S_2$, i.e. only above $\bar{\lambda} \in B_2$ with coincident pair and triple of points (see Remark 2.4 (2))

Notice that since $B_2 = \widetilde{E}_1 \cap E_2$, the proper transforms of $\widetilde{E}_1$, E_2 in V_3 cut the fiber of E_3 over any $\bar{\lambda} \in B_2$ in distinct points, say r_1, r_2. Fix now $\bar{\lambda} \in B_2$, i.e. a line λ with distinguished pair p_1, p_2 and triple q_1, q_2, q_3, and let $\mu \neq \lambda$ be a line. As a consequence of Lemma 2.5:

if μ touches the pair, then the line-condition in V_3 corresponding to μ contains r_2;

if μ touches the triple, then the line-condition in V_3 corresponding to μ contains r_1.

We conclude that the line-conditions can intersect over $\bar{\lambda}$ only if $p_1 = p_2 = q_1 = q_2 = q_3$, i.e. if $\bar{\lambda} \in B_2 \cap S_2$. □

Therefore in V_3 the line-conditions intersect along the smooth and irreducible 4-dimensional variety $S_3 \cong \mathrm{B}l_\Delta\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$. We choose S_3 as the center for the next blow-up: in other words, we let B_3 be S_3.

Note that $B_3 \cong \mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ has two natural projections onto $\check{\mathbb{P}}^2$: let l, m be the pull-backs via these projections of the hyperplane class in $\check{\mathbb{P}}^2$, and denote by e the exceptional divisor.

Theorem III (4). $B_3 \cong Bl_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

(i) *The intersection ring of* B_3 *is generated by* l, m, e, *and the relations* $em = el$, $l^3 = m^3 = 0$,

$$\int_{B_3} l^2 m^2 = 1, \qquad \int_{B_3} e^2 l^2 = -1,$$
$$\int_{B_3} e^3 l = -3, \qquad \int_{B_3} e^4 = -6;$$

(ii) $c(N_{B_3} V_3) = 1 + 7l + 17m - 16e + 126m^2 + 99lm + 21l^2 - 315el + 105e^2 + 582lm^2 + 237l^2m - 2517el^2 + 1611e^2l - 358e^3 + 1026l^2m^2 + 9174e^2l^2 - 3912e^3l + 652e^4$.

Proof. (i) Call k the hyperplane class in $\check{\mathbb{P}}^2 \cong \Delta \overset{\delta}{\hookrightarrow} \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$. Since $k = \delta^* l = \delta^* m$, then l, m, e generate the intersection ring of B_3 (cf. [F, Example 8.3.9]). $em = el$ is clear, while the other relations are checked observing that $e - e^2 + e^3 - e^4$ pushes forward to

$$s(\Delta, \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2) = \frac{1}{(1+k)^3}.$$

(ii) $j_3^* c(TV_3)$, $c(TB_3)$ can be obtained by applying the *blow-up Chern classes formula* (cf. [F, Theorem 15.4]).

Then $c(N_{B_3} V_3)$ is computed as $j_3^* c(TV_3)/c(TB_3)$. □

In the local coordinates given for V_2 in §3.2, equations for B_2 are

$$c_3 = 0, \qquad c_6 = 0$$

(recall that $B_2 = \widetilde{E}_1 \cap E_2$), thus we can choose coordinates $(d_1, \dots d_9)$ in an open set in V_3 such that

$$\begin{aligned} d_1 &= c_1, & d_2 &= c_2, & d_3 &= c_3, \\ d_4 &= c_4, & d_5 &= c_5, & d_6 d_3 &= c_6, \\ d_7 &= c_7, & d_8 &= c_8, & d_9 &= c_9. \end{aligned}$$

The equation of the exceptional divisor is $d_3 = 0$.

The map $\phi_2 \colon \mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \longrightarrow V_2$ lifts to a map $\phi_3 \colon \mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \longrightarrow V_3$, given in coordinates by

$$(\alpha_1, \alpha_2; u, t) \mapsto (3\alpha_1 + u, 3\alpha_2 + ut, u/2, 2t, t^2, -4, t, t^2, t^3).$$

Lemma 3.3. $j_3^* P_3 = l + 2m$, $j_3^* L_3 = 4l + 8m - 6e$; *the full intersection classes for points- and line-conditions with respect to* B_3 *are*

$$B_3 \circ P_3 = l + 2m \quad , \quad B_3 \circ L_3 = 1 + 4l + 8m - 6e.$$

Proof. $j_3^* P_3 = l + 2m$ because of the commutativity of the diagram

$$\begin{array}{ccc} B_3 = S_3 & \xrightarrow{j_3} & V_3 \\ \downarrow & & \downarrow \\ \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 & \xrightarrow{\phi_0} & V_0 \end{array},$$

by the definition of ϕ_0 in §3.0, and since P_3 is the pull-back of a hyperplane from V_0. $j_3^* L_3 = 4l + 8m - 6e$ because $L_3 = \pi_3^* L_2 - E_3$, $L_2 = \pi_2^* L_1 - E_2$, $L_1 = \pi_1^* L_0 - 2E_1$, L_0 is a hypersurface of degree 4 in V_0, and $\phi_1^* E_1 = 2e$ (Remark 1.4 (2)), $\phi_2^* E_2 = e$ (Remark 2.4 (1)), and $\phi_3^* E_3 = e$. No point-conditions in V_3 contain B_3, therefore $e_{B_3} P_3 = 0$; $e_{B_3} L_3 = 1$ follows from Lemma 0.1 (1), since V_0 and V_3 are isomorphic away from B_0 and from the exceptional divisors. □

Remarks 3.4.

(1) The equations of the line-conditions in V_3 corresponding to lines through the point $(1: 0: 0)$ are written in terms of $d_4, \dots, d_9$ only, as seen by direct computation.

(2) On the other hand, the last six coordinates $d_4, \dots, d_9$ of the image of a point via ϕ_3 are constant along divisors $\{t = \text{const.}\}$ in $\mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$.

Therefore, the behavior of line-conditions corresponding to lines containing $(1: 0: 0)$ is constant along the sets $\{t = \text{const.}\}$ in $\mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$. For example, to check the transversality of line-conditions corresponding to lines through $(1: 0: 0)$ at all points of the image of $\mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$, it is enough to check it for points not contained in e. This argument will be applied in §§3.4, 3.5; it can also be used to give a second proof of Proposition 3.2.

3.4. The fourth blow-up. Let $V_4 = \mathrm{B}l_{B_3} V_3$, write $\pi_4 \colon V_4 \longrightarrow V_3$ for the blow-up map, E_4 for the exceptional divisor, and denote by P_4, L_4 the proper transforms of P_3, L_3. Then $P_4 = \pi_4^* P_3$, and $L_4 = \pi_4^* L_3 - E_4$ (Lemma 0.1 (1)).

In V_4, the line-conditions will still intersect in a 4-dimensional smooth variety B_4, contained in E_4 (Proposition 4.1). B_4 will be the last center of blow-up.

Proposition 4.1. *The intersection of all line-conditions in V_4 is a smooth 4-dimensional subvariety B_4 of $E_4 = \mathbb{P}(N_{B_3} V_3)$. More precisely, $B_4 = \mathbb{P}(\mathcal{L})$, $\mathcal{L}$ a subline bundle of $N_{B_3} V_3$.*

Proof. First consider a point in $B_3 \cong \mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ not on the exceptional divisor e. There exist isomorphic neighborhoods of such a point in V_3 and of a point $\lambda\mu^2 \in S_0$, $\lambda \neq \mu$, in $V_0 = \mathbb{P}^9$; by Lemma 0.1 (3) the tangent hyperplanes to the line-conditions in V_0 at $\lambda\mu^2$ intersect in the 5-dimensional subspace

R_μ of V_0 consisting of the cubics containing μ. In fact, if λ, μ do not contain $(1:0:0)$ it is enough to consider line-conditions corresponding to lines containing $(1:0:0)$.

The tangent space to R_μ at $\lambda\mu^2$, $T_{\lambda\mu^2}R_\mu$, contains the 4-dimensional $T_{\lambda\mu^2}S_0$; as λ, μ vary, $T_{\lambda\mu^2}R_\mu / T_{\lambda\mu^2}S_0$ determine a line-bundle $\mathscr{L}^\circ$ over $S_0 - B_0 \cong B_3 - e$, and the intersection of the line-conditions in V_4 above points in B_3 outside e is supported on $\mathbb{P}(\mathscr{L}^\circ) \hookrightarrow \mathbb{P}(N_{B_3}V_3) = E_4$. By Remarks 3.4, $\mathscr{L}^\circ$ extends to a line bundle $\mathscr{L}$ over the whole B_3, and the line-conditions intersect along $\mathbb{P}(\mathscr{L})$ as claimed. □

We choose B_4 for the next (and last) center of blow-up: let $j_4 \colon B_4 \hookrightarrow V_4$ be the inclusion. The next lemma gives the information needed to compute $c(N_{B_4}V_4)$.

Lemma 4.2. $c_1(\mathscr{L}) = 3l + 3m - 4e$.

Proof. $\mathbb{P}(\mathscr{L})$ is isomorphic to $B_3 = \mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ via the projection map. To compute $c_1(\mathscr{L})$, notice that the restriction of $\mathscr{O}(-1)$ from $\mathbb{P}(N_{B_3}V_3) = E_4$ to $\mathbb{P}(\mathscr{L})$ is the pull-back of $\mathscr{L}$, so that, via the isomorphism $\mathbb{P}(\mathscr{L}) \longrightarrow B_3$, $c_1(\mathscr{L}) = j_4^* E_4$.

Consider then the divisor F_0 in V_0 with equation

$$\begin{vmatrix} 3a_0 & a_1 & a_2 \\ 2a_1 & 2a_3 & a_4 \\ 2a_2 & a_4 & 2a_5 \end{vmatrix} = 12a_0a_3a_5 - 3a_0a_4^2 - 4a_1^2a_5 + 4a_1a_2a_4 - 4a_2^2a_3 = 0.$$

The rows of the determinant are coefficients of second partial derivatives of the equation of a cubic, therefore it is clear that this divisor contains S_0 (the cubics in S_0 have a triple point). If F_1, F_2, F_3 denote the proper transforms of F_0 in V_1, V_2, V_3, one checks that $F_1 = \pi_1^* F_0 - 2E_1$, $F_2 = \pi_2^* F_1$, $F_3 = \pi_3^* F_2$. Since F_0 has degree 3, it follows that $j_3^* F_3 = 3l + 6m - 4e$. Now F_0 has multiplicity 1 along S_0: thus F_3 has multiplicity 1 along B_3, and if F_4 is the proper transform of F_3 in V_4 we get

$$c_1(\mathscr{L}) = 3l + 6m - 4e - j_4^* F_4.$$

By the description of $\mathscr{L}$ given in the proof of Proposition 4.1, F_4 meets $B_4 = \mathbb{P}(\mathscr{L})$ at a point mapping to $\lambda\mu^2 \in V_0 - B_0$ if the tangent hyperplane to F_0 at $\lambda\mu^2$ contains the space R_μ of cubics containing μ. Using this fact, one computes $j_4^* F_4 = 3m$, getting

$$c_1(\mathscr{L}) = 3l + 6m - 4e - 3m = 3l + 3m - 4e$$

as needed. □

Note that B_4 is isomorphic to $B_3 = \mathrm{B}l_\Delta \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ via the projection $\mathbb{P}(\mathscr{L}) \longrightarrow B_3$; thus its intersection ring is generated by the pull-backs of l, m, e, which we will still denote l, m, e, with the relations stated in Theorem III (4):

Theorem III (5). $B_4 \cong B_3$.

(i) *The intersection ring of* B_4 *is generated by* l, m, e, *and* $em = el$, $l^3 = m^3 = 0$,

$$\int_{B_3} l^2 m^2 = 1, \qquad \int_{B_3} e^2 l^2 = -1,$$
$$\int_{B_3} e^3 l = -3, \qquad \int_{B_3} e^4 = -6;$$

(ii) $c(N_{B_4}V_4) = 1 - 5l + 5m + 18m^2 - 27lm + 3l^2 + 21el - 7e^2 - 30lm^2 + 75l^2m - 225el^2 + 135e^2l - 30e^3 + 75l^2m^2$.

Proof. (i) is noticed above.

(ii) The Euler sequence

$$0 \longrightarrow \mathscr{O}_{E_4} \longrightarrow N_{B_3}V_3 \otimes \mathscr{O}(1) \longrightarrow T_{E_4|B_3} \longrightarrow 0$$

restricts to

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}(\mathscr{L})} \longrightarrow N_{B_3}V_3 \otimes \check{\mathscr{L}} \longrightarrow T_{E_4|B_3} \longrightarrow 0$$

on $\mathbf{P}(\mathscr{L})$ (for ease of reading, we have suppressed the pull-back signs). Since $B_4 = \mathbf{P}(\mathscr{L}) \cong B_3$ via the projection, it follows

$$c(N_{B_4}E_4) = c(T_{E_4|B_3}) = c(N_{B_3}V_3 \otimes \check{\mathscr{L}}),$$

so that

$$c(N_{B_4}V_4) = j_4^* c(N_{E_4}V_4) c(N_{B_4}E_4) = c(\mathscr{L}) c(N_{B_3}V_3 \otimes \check{\mathscr{L}}),$$

and (ii) follows. □

Lemma 4.3. $j_4^* P_4 = l + 2m$, $j_4^* L_4 = l + 5m - 2e$. *The full intersection classes for the point- and line-conditions with respect to* B_4 *are*

$$B_4 \circ P_4 = l + 2m, \qquad B_4 \circ L_4 = 1 + l + 5m - 2e.$$

Proof. $P_4 = \pi_4^* P_3$ implies $j_4^* P_4 = l + 2m$. The restriction of $\mathscr{O}(-1)$ from $E_4 = \mathbf{P}(N_{B_3}V_3)$ to $\mathbf{P}(\mathscr{L})$ is the pull-back of $\mathscr{L}$, so $j_4^* L_4 = j_4^*(\pi_4^* L_3 - E_4) = 4l + 8m - 6e - c_1(\mathscr{L}) = l + 5m - 2e$ (Lemma 4.2). The point-conditions in V_4 do not contain B_4; the line-conditions in V_4 are generically smooth along B_4 since the line-conditions in V_3 are generically smooth along B_3. □

3.5. The fifth blow-up. Let $V_5 = \mathrm{Bl}_{B_4} V_4$, write $\pi_5 \colon V_5 \longrightarrow V_4$ for the blow-up map, E_5 for the exceptional divisor, denote by $\widetilde{E}_4$, P_5, L_5 the proper transforms of E_4, P_4, L_4.

Finally, we will see that the line-conditions 'separate' in V_5 (Proposition 5.3), concluding the proof of Theorem III.

Consider $\widetilde{E}_4 \cap E_5 = \mathbf{P}(N_{B_4}E_4)$.

Denote by $\mathscr{O}_1(-1)$ (resp. $\mathscr{O}_2(-1)$) the pull-back of the universal line-bundle from the first (resp. second) factor of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ to $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$. Recalling that $B_3 - e \cong S_0 - B_0 \hookrightarrow \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ (and omitting pull-backs for sake of notations)

$$N_{B_3-e}V_3 \cong N_{S_0-B_0}V_0 = T\mathbb{P}^9/T\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2;$$

if $\mathbb{P}^2 = \mathbb{P}Q$, so that $\mathbb{P}^9 = \mathbb{P}(\mathrm{Sym}^3 \check{Q})$, then $T\mathbb{P}^9$ is given by the Euler sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^9} \longrightarrow \mathrm{Sym}^3 \check{Q} \otimes \mathscr{O}_{\mathbb{P}^9}(1) \longrightarrow T\mathbb{P}^9 \longrightarrow 0,$$

thus

$$(*) \qquad \begin{aligned} N_{B_3-e}V_3 &\cong (\mathrm{Sym}^3 \check{Q} \otimes \mathscr{O}_{\mathbb{P}^9}(1)/\mathscr{O}_{\mathbb{P}^9})/T\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\ &\cong (\mathrm{Sym}^3 \check{Q} \otimes \mathscr{O}_1(1) \otimes \mathscr{O}_2(2)/\mathscr{O}_{B_3-e})/T\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2. \end{aligned}$$

$N_{B_4}E_4 \cong T_{E_4|B_4}$ is given by

$$0 \longrightarrow \mathscr{O}_{E_4} \longrightarrow N_{B_3}V_3 \otimes \mathscr{O}_{E_4}(1) \longrightarrow T_{E_4|B_4} \longrightarrow 0,$$

restricting on $B_4 = \mathbb{P}(\mathscr{L})$ to

$$0 \longrightarrow \mathscr{O}_{B_4} \longrightarrow N_{B_3}V_3 \otimes \check{\mathscr{L}} \longrightarrow N_{B_4}E_4 \longrightarrow 0.$$

On the other hand, over $B_3 - e$ the line bundle $\mathscr{L}$ restricts to

$$(**) \qquad \begin{aligned} \mathring{\mathscr{L}} &\cong (\mathrm{Sym}^2 \check{Q} \otimes \mathscr{O}_{\mathbb{P}^5}(1)/\mathscr{O}_{\mathbb{P}^5})/T\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \\ &\cong (\mathrm{Sym}^2 \check{Q} \otimes \mathscr{O}_1(1) \otimes \mathscr{O}_2(1)/\mathscr{O}_{B_3-e})/T\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \end{aligned}$$

(this follows from the description of $\mathring{\mathscr{L}}$ given in the proof of Proposition 4.1). By $(*)$ and $(**)$,

$$N_{B_4}E_4 \cong (N_{B_3}V_3 \otimes \check{\mathscr{L}})/\mathscr{O}_{B_4} \cong (N_{B_3}V_3/\mathscr{L}) \otimes \check{\mathscr{L}}$$

restricts over $B_4 - e$ to

$$\frac{\mathrm{Sym}^3 \check{Q}}{\mathrm{Sym}^2 \check{Q} \otimes \mathscr{O}_2(-1)} \otimes \mathscr{O}_1(1) \otimes \mathscr{O}_2(2) \otimes \check{\mathscr{L}}.$$

Therefore, over a point in $B_4 - e$, mapping to $\lambda\mu^2 \in S_0 - \Delta$, the fiber of $N_{B_4}E_4$ can be identified with the space $\mathrm{Sym}^3 \check{Q}/\mathrm{Sym}^2 \check{Q}$, where the inclusion $\mathrm{Sym}^2 \check{Q} \hookrightarrow \mathrm{Sym}^3 \check{Q}$ is given by the multiplication by μ. This space is canonically isomorphic to the space of homogeneous degree-3 polynomials on the line with equation $\mu = 0$; consequently, a point of $\widetilde{E}_4 \cap E_5 = \mathbb{P}(N_{B_4}E_4)$ over $\lambda\mu^2$, $\lambda \neq \mu$, can be pictured as a cubic consisting of a line and a double line, with three distinguished points on the double line.

For $\lambda \neq \mu$ lines in $\mathbb{P}^2$, consider a point $\overline{\lambda\mu^2} \in \widetilde{E}_4 \cap E_5$, i.e. $\lambda\mu^2$ 'with a distinguished triple of points specified on μ', and a line $\nu \subset \mathbb{P}^2$.

Lemma 5.1. *Suppose* ν *does not contain* $\lambda \cap \mu$. *Then the line-condition in* V_5 *corresponding to* ν *contains* $\overline{\lambda\mu^2}$ *if and only if* ν *contains a point of the triple on* μ.

Proof. Let L_i be the line-condition in V_i corresponding to ν. Since $\nu \not\supset \lambda\cap\mu$, then L_0 is nonsingular at $\lambda\mu^2$ by Lemma 0.1 (1). Let $H_p \subset \mathrm{Sym}^3 \check{Q}$ denote the space of cubic polynomials on Q vanishing at p. As $\lambda\mu^2$ varies in S_0, the $H_{\nu\cap\mu}$ define a subbundle $\mathscr{H}$ of $\mathrm{Sym}^3 \check{Q}$ over a neighborhood of $\lambda\mu^2$; notice that $\mathscr{H} \supset \mathrm{Sym}^2 \check{Q} \otimes \mathscr{O}_2(-1)$. By Lemma 0.1 (3), the tangent space to L_0 at $\lambda\mu^2$ is contained in $T_{\lambda\mu^2}\mathbb{P}^9$ as the image of $\mathscr{H} \otimes \mathscr{O}_{\mathbb{P}^9}(1)$; tracing the argument preceding this lemma identifies then the fiber of $L_5 \cap \widetilde{E}_4 \cap E_5$ over $\lambda\mu^2$ with $\mathbb{P}(H_{\nu\cap\mu}/\mathrm{Sym}^2 \check{Q})$, the space of triples on μ touching ν. □

Remark 5.2. Notice that this lemma implies that the intersection of all line-conditions in V_5 must be disjoint from $\widetilde{E}_4 \cap E_5$. We will prove that it is empty by showing that it must also *be contained* in $\widetilde{E}_4 \cap E_5$.

Proposition 5.3. *The intersection of all line-conditions in* V_5 *is empty.*

Proof. The line-conditions can intersect only in E_5. By Remarks 3.4, it is enough to check that the intersection is empty above $B_4 - e$; and since the matter is invariant under projective transfomations, it is enough to check that the intersection of all line-conditions in V_5 is empty over a single point $\lambda\mu^2 \in B_4$, with $\lambda \neq \mu$.

The fiber $(E_5)_{\lambda\mu^2}$ of $E_5 = \mathbb{P}(N_{B_4} V_4)$ over $\lambda\mu^2$ is a 4-dimensional projective space $\mathbb{P}^4$. For $\nu \not\supset \lambda \cap \mu$, the association

$$\nu \text{ line in } \mathbb{P}^2 \mapsto (E_5)_{\lambda\mu^2} \cap \text{ line-condition in } V_5 \text{ corresponding to } \nu$$

determines a rational map $\check{\mathbb{P}}^2 \dashrightarrow \check{\mathbb{P}}^4$. Notice that by the nonsingularity of $\check{\mathbb{P}}^2$, this extends in codimension 1, so it must be defined for at least all $\nu \neq \lambda, \mu$. Let then $\nu \supset \lambda\cap\mu$, $\nu \neq \lambda, \mu$, denote by $\lambda\mu^2$ also the point on B_4 over $\lambda\mu^2 \in \mathbb{P}^9$, and write $L_{\nu'}$ for the line-condition in V_4 corresponding to a line $\nu' \not\supset \lambda \cap \mu$. A coordinate computation shows that as ν' approaches ν, its tangent space at $\lambda\mu^2$ $T_{\lambda\mu^2}L_{\nu'}$ approaches $T_{\lambda\mu^2}E_4$ (as subspaces of $T_{\lambda\mu^2}V_4$). It follows that the image of ν under $\check{\mathbb{P}}^2 \dashrightarrow \check{\mathbb{P}}^4$ is the intersection of $\widetilde{E}_4$ with the fiber of E_5: this implies that the intersection of all line-conditions in V_5 is included in $\widetilde{E}_4 \cap E_5$. On the other hand, by Remark 5.2 the intersection must be disjoint from $\widetilde{E}_4 \cap E_5$: therefore, it must be empty. □

A different proof of an equivalent statement can be found in [St, II, p. 146]. Proposition 5.3 concludes the proof of Theorem III: $\widetilde{V} = V_5$ is a smooth variety of complete cubics. By Corollary I, the number of smooth cubics containing

n_p points and tangent to n_l lines in general position $(n_p + n_l = 9)$ is then $\int_{V_5} P_5^{n_p} L_5^{n_l}$. In the next section we will apply Theorem II to compute these intersection numbers.

4 Computation of the characteristic numbers

We work over an algebraically closed field of characteristic $\neq 2, 3$. The notations for this section are those used in the statement of Theorem III: $V_0 = \mathbb{P}^9$, V_i is the ith blow-up, B_i the center for the $i+1$th blow-up, the intersection rings of the B_i's are generated by various subsets of $\{h, \varepsilon, \phi, l, m, e\}$, with the relations listed in Theorem III. Furthermore (as in the rest of §3) P_i, L_i denote respectively the point- and line-conditions in V_i; we found in §3 that the full intersection classes of P_i, L_i with respect to B_i, $i = 0, \dots, 4$, are respectively

$$\begin{aligned} B_0 \circ P_0 &= 3h, & B_0 \circ L_0 &= 2 + 12h, \\ B_1 \circ P_1 &= 3h, & B_1 \circ L_1 &= 1 + 12h - 2\varepsilon, \\ B_2 \circ P_2 &= 3h, & B_2 \circ L_2 &= 1 + 12h - 2\varepsilon - \varphi, \\ B_3 \circ P_3 &= l + 2m, & B_3 \circ L_3 &= 1 + 4l + 8m - 6e, \\ B_4 \circ P_4 &= l + 2m, & B_4 \circ L_4 &= 1 + l + 5m - 2e. \end{aligned}$$

Also, Theorem III lists the total Chern classes $c(N_{B_i} V_i)$ and the relations in dimension 0 in the Chow groups of the B_i's. Therefore, the following statement translates the computation of the characteristic numbers of a family F into the computation of a degree and of five full intersection classes $B_i \circ F_i$:

Theorem IV (Notations of Theorem III). *Let F be an r-dimensional subvariety in $\mathbb{P}^9$ parametrizing a family of reduced cubics, and let F_i be the proper transform in V_i of the closure F_0 of F. Also, let f be the degree of the closure of F. Then the number $N_F(n_p P, n_l L)$ of elements (counted with multiplicities) of F containing n_p given points and tangent to n_l given lines in general position, with $n_p + n_l = r$, is*

$$N_F(n_p P, n_l L) = 4^{n_l} \cdot f - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ P_i)^{n_p} (B_i \circ L_i)^{n_l} (B_i \circ F_i)}{c(N_{B_i} V_i)}.$$

Furthermore, the elements containing the given points and properly tangent to the given lines are counted with multiplicity 1.

Proof. This follows from

(1) $$\int_{V_0} P_0^{n_p} L_0^{n_l} F_0 = 4^{n_l} \cdot f,$$

(2) $$\int_{V_{i+1}} P_{i+1}^{n_p} L_{i+1}^{n_l} F_{i+1} = \int_{V_i} P_i^{n_p} L_i^{n_l} F_i - \int_{B_i} \frac{(B_i \circ P_i)^{n_p} (B_i \circ L_i)^{n_l} (B_i \circ F_i)}{c(N_{B_i} V_i)},$$

(3) $$N_F(n_p P, n_l L) = \int_{V_5} P_5^{n_p} L_5^{n_l} F_5.$$

(1) follows from Bézout's Theorem, (2) from Theorem II, and (3) from Theorems I and III. □

For the family F of smooth cubics, we have $F_i = V_i$, so that $B_i \circ F_i = [B_i]$. Also, all tangencies are proper, thus the numbers given by Theorem IV are in fact the 'characteristic numbers'. Writing $N(n_pP, n_lL) = N_F(n_pP, n_lL)$ in this case, we get

Corollary IV. *The characteristic numbers for the family of smooth plane cubics are given by*

$$N(n_pP, n_lL) = \begin{cases} 1, & n_p = 9,\ n_l = 0, \\ 4, & n_p = 8,\ n_l = 1, \\ 16, & n_p = 7,\ n_l = 2, \\ 64, & n_p = 6,\ n_l = 3, \\ 256, & n_p = 5,\ n_l = 4, \\ 976, & n_p = 4,\ n_l = 5, \\ 3424, & n_p = 3,\ n_l = 6, \\ 9766, & n_p = 2,\ n_l = 7, \\ 21004, & n_p = 1,\ n_l = 8, \\ 33616, & n_p = 0,\ n_l = 9. \end{cases}$$

Proof. Theorem IV gives

$$(*) \qquad N(n_pP, n_lL) = 4^{n_l} - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ P_i)^{n_p}(B_i \circ L_i)^{n_l}}{c(N_{B_i}V_i)};$$

listing only the nonzero contributions, and understanding $n_p = 9 - n_l$:

$$\int_{B_0} \frac{(3h)^{n_p}(2 + 12h)^{n_l}(1 + h)^3}{(1 + 3h)^{10}} = \begin{cases} 1152, & n_l = 7, \\ 16128, & n_l = 8, \\ 125952, & n_l = 9, \end{cases}$$

$$\int_{B_1} \frac{(3h)^{n_p}(1 + 12h - 2\varepsilon)^{n_l}(1 + 2h - \varepsilon)^6}{(1 + \varepsilon)(1 + 3h - \varepsilon)^{10}} = \begin{cases} 441, & n_l = 7, \\ 5229, & n_l = 8, \\ 32214, & n_l = 9, \end{cases}$$

$$\int_{B_2} \frac{(3h)^{n_p}(1 + 12h - 2\varepsilon - \varphi)^{n_l}}{(1 + \varphi)(1 + \varepsilon - \varphi)} = \begin{cases} 2295, & n_l = 7, \\ 21411, & n_l = 8, \\ 97146, & n_l = 9, \end{cases}$$

$$\int_{B_3} \frac{(l+2m)^{n_p}(1+4l+8m-6e)^{n_l}}{(1+7l+17m-16e+\cdots)} = \begin{cases} 24, & n_l = 5, \\ 390, & n_l = 6, \\ 1572, & n_l = 7, \\ 18, & n_l = 8, \\ -22635, & n_l = 9, \end{cases}$$

$$\int_{B_4} \frac{(l+2m)^{n_p}(1+l+5m-2e)^{n_l}}{(1-5l+5m+\cdots)} = \begin{cases} 24, & n_l = 5, \\ 282, & n_l = 6, \\ 1158, & n_l = 7, \\ 1746, & n_l = 8, \\ -4149, & n_l = 9, \end{cases}$$

Each of these computations is performed by extracting the 0th dimensional terms in the series and using the relations in the rings of the B_i's listed in Theorem III. For example:

$$\begin{aligned} &\int_{B_4} \frac{(1+l+5m-2e)^9}{(1-5l+5m+\ldots)} \\ &= \int_{B_4} 48654 l^2 m^2 + 126129 e^2 l^2 - 29508 e^3 l + 2533 e^4 \\ &= 48654 \cdot 1 + 126129 \cdot (-1) - 29508 \cdot (-3) + 2533 \cdot (-6) \\ &= -4149. \end{aligned}$$

The computations were carried out using Macsyma.

These results and $(*)$ above give $N(n_p P, n_l L) = 4^{n_l}$ for $n_l = 0, \ldots, 4$ and $n_p = 9 - n_l$, and

$$N(n_p P, n_l L) = \begin{cases} 1024 - 0 - 0 - 0 - 24 - 24 = 976, \\ 4096 - 0 - 0 - 0 - 390 - 282 = 3424, \\ 16384 - 1152 - 441 - 2295 - 1572 - 1158 = 9766, \\ 65536 - 16128 - 5229 - 21311 - 18 - 1746 = 21004, \\ 262144 - 125952 - 32214 - 97146 + 22635 + 4149 = 33616 \end{cases}$$

for $n_l = 5, \ldots, 9$ as stated. □

Corollary IV agrees with Maillard and Zeuthen's result. More generally, the relevant information needed to apply Theorem IV to a family F is the behavior of the proper transforms F_i of the closure F_0 of F, with respect to the B_i's. For example, if F_0 is a *divisor* of $\mathbb{P}^9$, all one needs is the degree of F_0 and the five multiplicities of the F_i along the B_i (for F_i divisors, this information gives $(B_i \circ F_i)$). For example, for F_0 the divisor of *singular* cubics the multiplicities are 8,5,3,6,6, as we will compute in a forthcoming note. These multiplicities and the degree of F_0 (12) are enough to compute the $N_F(n_p P, n_l L)$ for nodal cubics. From them, the characteristic numbers for nodal cubics will be obtained by further applications of Theorem IV to the families of nodal cubics with node on given line and node at a given point.

5 A CODIMENSION-2 CONDITION

Maillard and Zeuthen's results for smooth cubics go further than Corollary IV above. After computing the characteristic numbers involving point- and line-conditions, they list the numbers also involving the codimension-2 conditions expressing *tangency to a line at a given point.*

Such conditions are linear: in a sense they are the intersection of two 'infinitely near' point-conditions. The numbers reflect this fact by agreeing with appropriate characteristic numbers from Corollary IV for low n_l; but as n_l grows larger and nonreduced curves enter into the picture, their position with respect to the flag becomes relevant and one expects discrepancies to occur. It is natural to inquire whether the information we need to apply Theorem IV to the computation of the numbers involving codimension-1 conditions is enough to obtain these other results; this is indeed the case, as we will show in this section.

The geometry of the situation is captured in five full intersection classes (Proposition 5.1); once they are computed, a statement analogous to Theorem IV gives the numbers involving these codimension-2 conditions for a family F if the classes $B_i \circ F_i$ are known. As in §4, the application to the family of smooth plane cubics (over an algebraically closed field of characteristic $\neq 2, 3$) is then immediate.

We will keep the style of the notations introduced in §1: call *point-line-conditions* M the linear subspaces $\mathbb{P}^{N-2} \hookrightarrow \mathbb{P}^N$ formed by the plane curves tangent to a given line at a given point; for any variety $\widetilde{V}$ mapping to $\mathbb{P}^N$, isomorphically over $\mathbb{P}^N - S$, call *point-line-conditions in* $\widetilde{V}$ the proper transforms $\widetilde{M}$ of the conditions M of $\mathbb{P}^N$. $\widetilde{M}$ is regularly imbedded outside the inverse image of S in $\widetilde{V}$; therefore, if the intersection of $\widetilde{M}$ with a subvariety $\widetilde{F}$ of $\widetilde{V}$ is proper and has no components lying over S, then the product $\widetilde{M} \cdot \widetilde{F}$ is defined.

Theorem I$'$. *Let $\widetilde{V}$ be a variety of complete curves of degree d, F an r-dimensional subvariety in $\mathbb{P}^N$ parametrizing a family of reduced curves, and let $\widetilde{F}$ be the proper transform in $\widetilde{V}$ of the closure of F. Then the number of elements (counted with multiplicities) of F containing n_p given points, tangent to n_l given lines, and tangent to n_m given lines at specified points (all choices being general), with $n_p + n_l + 2n_m = r$, is $\widetilde{P}^{n_p} \cdot \widetilde{L}^{n_l} \cdot \widetilde{M}^{n_m} \cdot \widetilde{F}$. Furthermore, the elements satisfying the conditions and properly tangent to the lines are counted with multiplicity one.*

Proof. We just sketch the arguments here, since they closely resemble those in §1. We also assume the notations and the basic set-up from §1. The main observation is the analogous for point-line-conditions of Lemma 1 in §1, namely:

Claim. *For* $F \subset \mathbb{P}^N$, *there exists a point-line-condition* M *such that* $\widetilde{\overline{M \cap F}} = \widetilde{M} \cap \widetilde{F}$.

Indeed, one has to check that $\widetilde{M} \cap \widetilde{F}$ does not have components over S. But a point-line-condition M is contained in the intersection of the corresponding point-condition P and line-condition L, so that $\widetilde{M} \cap \widetilde{F} \subset \widetilde{L} \cap \widetilde{P} \cap \widetilde{F}$. We can choose the point so that $\widetilde{P} \cap \widetilde{F}$ has no components over S (Lemma 2 in §1), and for a general line through that point we can get $\widetilde{L} \cap \widetilde{P} \cap \widetilde{M}$ with no components over S (the set of line-conditions corresponding to lines through a point is nondegenerate in $\check{\mathbb{P}}^M$).

The claim implies the first part of the theorem, by the same argument in the proof of Theorem I (1) in §1.

The proof of the statement about multiplicities is likewise similar to the proof of Theorem I (2) in §1. □

Before stating Theorem IV$'$, we compute the full intersection classes $B_i \circ M_i$, $i = 0, \dots, 4$, for point-line-conditions. Here the notations are those used in Theorem III, M_0 denotes a point-line-condition in $V_0 = \mathbb{P}^9$, and M_i is the proper-transform of M_{i-1} in V_i (i.e., a 'point-line-condition' in V_i).

Proposition 5.1 (Full intersection classes for point-line-conditions).

(1) $B_0 \circ M_0 = 2h + 9h^2$,
(2) $B_1 \circ M_1 = h + 9h^2 - 2\varepsilon h$,
(3) $B_2 \circ M_2 = h + 9h^2 - 2\varepsilon h - \phi h$,
(4) $B_3 \circ M_3 = m + l^2 + 4lm + 4m^2 - 6el$,
(5) $B_4 \circ M_4 = m + l^2 + lm + m^2 - 2el$.

Proof. The main tools are the geometry of the blow-ups (§3), and (iii) from §2.

(1) M_0 is nonsingular, has codimension 2 and intersects B_0 along the pencil $\mathbb{P}^1 \subset \mathbb{P}^2 = B_0$ of triple lines through the given point; therefore $B_0 \circ M_0 = [B_0 \cap M_0] + B_0 \cdot M_0$. An algebraic check gives $[B_0 \cap M_0] = 2h$; and since the hyperplane in $\mathbb{P}^9 = V_0$ pulls-back to $3h$ on B_0, (1) follows.

(2) Notice that M_0 is contained in the line-condition L_0 corresponding to the given line, and in the point-condition P_0 corresponding to the point. The fiber of M_1 over a point of $B_0 \cap M_0$ is 5-dimensional and contained into (therefore coinciding with) the irreducible 5-dimensional fiber of L_1 over the same point. It follows that $B_1 \cap M_1$ is (set-theoretically) the fiber in B_1 of $B_0 \cap M_0$. Also, $\dim(\mathrm{Sing} M_1) \le \dim B_0 \cap M_0 = 1$, thus M_1 is generically nonsingular along $B_1 \cap M_1$; it follows $B_1 \circ M_1 = [B_1 \cap M_1] + B_1 \cdot M_1$. Since $M_1 \subset L_1 \cap P_1$, L_1 is generically nonsingular along B_1 (Lemma 1.5 (2) in §3), and P_1 cuts transversally B_1, then $[B_1 \cap M_1] = h$. Finally, one applies Fulton's blow-up formula (Theorem 6.7 in [F]) to get $B_1 \cdot M_1 = 9h^2 - 2\varepsilon h$, as stated.

(3), (4) and (5) are obtained using the same arguments. □

Now we can state the extension of Theorem IV:

Theorem IV$'$ (Notations of Theorem III). *Let F an r-dimensional subvariety in $\mathbf{P}^9$ parametrizing a family of reduced cubics, and let F_i be the proper transform in V_i of the closure F_0 of F. Also, let f be the degree of the closure of F. Then the number $N_F(n_pP, n_lL, n_mM)$ of elements (counted with multiplicities) of F containing n_p given points, tangent to n_l given lines, and tangent to n_m given lines at specified points (all choices being general), with $n_p + n_l + 2n_m = r$, is*

$$N_F(n_pP, n_lL, n_mM) = 4^{n_l} \cdot f - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ P_i)^{n_p}(B_i \circ L_i)^{n_l}(B_i \circ M_i)^{n_m}(B_i \circ F_i)}{c(N_{B_i}V_i)}.$$

Furthermore, the elements containing the given points and properly tangent to the given lines are counted with multiplicity one.

Proof. Similarly to Theorem IV, this is a consequence of

$$\int_{V_0} P_0^{n_p} L_0^{n_l} M_0^{n_m} F_0 = 4^{n_l} \cdot f, \tag{1}$$

$$\int_{V_{i+1}} P_{i+1}^{n_p} L_{i+1}^{n_l} M_{i+1}^{n_m} F_{i+1} = \int_{V_i} P_i^{n_p} L_i^{n_l} M_i^{n_m} F_i - \int_{B_i} \frac{(B_i \circ P_i)^{n_p}(B_i \circ L_i)^{n_l}(B_i \circ M_i)^{n_m}(B_i \circ F_i)}{c(N_{B_i}V_i)}, \tag{2}$$

$$N_F(n_pP, n_lL) = \int_{V_5} P_5^{n_p} L_5^{n_l} M_5^{n_m} F_5, \tag{3}$$

where now (3) follows from Theorem I$'$ and III. □

This applies immediately to the family of smooth cubics; denoting the numbers in this case by $N(n_pP, n_lL, n_mM)$:

Corollary IV$'$.

$$N(n_pP, n_lL, 1M) = \begin{cases} 1, & n_p = 7,\ n_l = 0, \\ 4, & n_p = 6,\ n_l = 1, \\ 16, & n_p = 5,\ n_l = 2, \\ 64, & n_p = 4,\ n_l = 3, \\ 244, & n_p = 3,\ n_l = 4, \\ 856, & n_p = 2,\ n_l = 5, \\ 2344, & n_p = 1,\ n_l = 6, \\ 4726, & n_p = 0,\ n_l = 7; \end{cases}$$

$$N(n_pP, n_lL, 2M) = \begin{cases} 1, & n_p = 5,\ n_l = 0, \\ 4, & n_p = 4,\ n_l = 1, \\ 16, & n_p = 3,\ n_l = 2, \\ 62, & n_p = 2,\ n_l = 3, \\ 220, & n_p = 1,\ n_l = 4, \\ 576, & n_p = 0,\ n_l = 5; \end{cases}$$

$$N(n_pP, n_lL, 3M) = \begin{cases} 1, & n_p = 3,\ n_l = 0, \\ 4, & n_p = 2,\ n_l = 1, \\ 16, & n_p = 1,\ n_l = 2, \\ 58, & n_p = 0,\ n_l = 3; \end{cases}$$

$$N(n_pP, n_lL, 4M) = \begin{cases} 1, & n_p = 1,\ n_l = 0, \\ 4, & n_p = 0,\ n_l = 1. \end{cases}$$

Proof. This follows from Theorem IV$'$ applied to the family of *smooth* cubics; in this case, $B_i \circ F_i = [B_i]$. We just list here the relevant contributions:

$$N((7-n_l)P, n_lL, 1M) = \begin{cases} 256 - 0 - 0 - 0 - 6 - 6 = 244, & n_l = 4, \\ 1024 - 0 - 0 - 0 - 99 - 69 = 856, & n_l = 5, \\ 4096 - 384 - 147 - 765 - 240 - 216 = 2344, & n_l = 6, \\ 16384 - 4992 - 1596 - 6372 + 1287 + 15 = 4726, & n_l = 7; \end{cases}$$

$$N((5-n_l)P, n_lL, 2M) = \begin{cases} 64 - 0 - 0 - 0 - 1 - 1 = 62, & n_l = 3, \\ 256 - 0 - 0 - 0 - 21 - 15 = 220, & n_l = 4, \\ 1024 - 128 - 49 - 255 + 13 - 29 = 576, & n_l = 5; \end{cases}$$

$$N(0P, 3L, 3M) = 64 - 0 - 0 - 0 - 3 - 3 = 58. \quad \square$$

Corollary IV$'$ also agrees with Maillard and Zeuthen's results.

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